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Multiple existence results of solutions for the Neumann problems via super- and sub-solutions

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Abstract

By variational methods, we provide existence results of multiple solutions for quasilinear elliptic equations under the Neumann boundary condition. Our main result shows the existence of two constant sign solutions and a sign changing solution in the case where we do not impose the subcritical growth condition to the nonlinear term not including derivatives of the unknown function. The studied equations contain the p-Laplacian problems as a special case. Moreover, we give a result concerning a local minimizer in $C^1(\overline{\Omega})$ versus $W^{1,p}(\Omega)$. Auxiliary results of independent interest are also obtained: a density property for the space $W^{1,p}(\Omega)$, a strong maximum principle of Zhang's type, and a Moser's iteration scheme depending on a parameter.

Keywords: Quasilinear elliptic equations; Neumann boundary value condition; Super-solution and sub-solution for the Neumann problems; Mountain pass theorem; Second deformation lemma

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1. Introduction and statements of main results

In this paper, we consider the existence of non-trivial multiple solutions for the following quasilinear elliptic equation

$$\begin{cases} -\operatorname{div} A(x, \nabla u) = f(x, u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2 boundary $\partial\Omega$ and n denotes the outward unit normal vector on $\partial\Omega$. Here, $A : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ is a map which is strictly monotone in the second variable and satisfies certain regularity conditions (see the following assumption (A)). Eq. (P) contains the corresponding p-Laplacian problem as a special case. However, in general, we do not suppose that this operator is $(p-1)$-homogeneous in the second variable.

Throughout this paper, we assume that the map A and the nonlinear term f satisfy the following assumptions (A) *and* (f)*, respectively*:

(A) $A(x, y) = a(x, |y|)y$, where $a(x, t) > 0$ for all $(x, t) \in \overline{\Omega} \times (0, +\infty)$, $1 < p < \infty$ and
  (i) $A \in C^0(\overline{\Omega} \times \mathbb{R}^N, \mathbb{R}^N) \cap C^1(\overline{\Omega} \times (\mathbb{R}^N \setminus \{0\}), \mathbb{R}^N)$;
  (ii) there exists $C_1 > 0$ such that

$$\left|D_y A(x, y)\right| \leqslant C_1 |y|^{p-2} \quad \text{for every } x \in \overline{\Omega}, \text{ and } y \in \mathbb{R}^N \setminus \{0\};$$

  (iii) there exists $C_0 > 0$ such that

$$D_y A(x, y)\xi \cdot \xi \geqslant C_0 |y|^{p-2} |\xi|^2 \quad \text{for every } x \in \overline{\Omega}, \ y \in \mathbb{R}^N \setminus \{0\} \text{ and } \xi \in \mathbb{R}^N;$$

  (iv) there exists $C_2 > 0$ such that

$$\left|D_x A(x, y)\right| \leqslant C_2 \left(1 + |y|^{p-1}\right) \quad \text{for every } x \in \overline{\Omega}, \ y \in \mathbb{R}^N \setminus \{0\};$$

  (v) there exist $C_3 > 0$ and $1 \geqslant t_0 > 0$ such that

$$\left|D_x A(x, y)\right| \leqslant C_3 |y|^{p-1} \left(-\log|y|\right)$$

  for every $x \in \overline{\Omega}$, $y \in \mathbb{R}^N$ with $0 < |y| < t_0$.

(f) f is a Carathéodory function on $\Omega \times \mathbb{R}$ with $f(x, 0) = 0$ for a.e. $x \in \Omega$ and f is bounded on bounded sets.

In this paper, we say that $u \in W^{1,p}(\Omega)$ is a (weak) solution of (P) if

$$\int_\Omega A(x, \nabla u)\nabla\varphi \, dx = \int_\Omega f(x, u)\varphi \, dx$$

for all $\varphi \in W^{1,p}(\Omega)$ provided the integral on the right-hand side exists. And we say that u is a positive (resp. negative) solution of (P) if $u \in W^{1,p}(\Omega)$ is a solution in the above sense and $u(x) > 0$ (resp. $u(x) < 0$) for a.e. $x \in \Omega$.

The hypothesis (A) is considered in the study of quasilinear elliptic problems (cf. [17, Example 2.2.] and see also [13,16,21]). In particular, for $A(x,y) = |y|^{p-2}y$, that is, $\operatorname{div} A(x,\nabla u)$ stands for the usual p-Laplacian $\Delta_p u$, we can take $C_0 = C_1 = p-1$ in (A). Conversely, in the case where $C_0 = C_1 = p-1$ holds in (A), by the inequalities in Remark 7(ii) and (iii) in Section 2, we see $a(x,t) = |t|^{p-2}$ whence $A(x,y) = |y|^{p-2}y$.

Many authors have considered the existence of solutions for the quasilinear elliptic equations by variational methods in the subcritical case (see (g) in Section 2) under the Neumann boundary condition (cf. [1,4,7,16,17,19,20]). On the other hand, it is generally hard to handle the quasilinear elliptic equations by variational methods without the subcritical growth condition. In [9], Carl and the second author proved the existence of multiple solutions for the p-Laplace equation under the *Dirichlet boundary condition* without the subcritical growth condition on f.

We point out that our functional setting for problem (P) under Neumann boundary condition is essentially different from that for the Dirichlet case treated in [9]. Namely we should use a variational approach suitable for the specific character of the Neumann problems. For instance, contrary to the Dirichlet case, constants can be taken as test functions whereas the Poincaré inequality does not hold. Actually, the study of the nonlinear Neumann problem (P) is based on newly established results regarding local minimizers, regularity, strong maximum principle that are distinct from those used for dealing with the Dirichlet problem in [9].

The main purpose of this paper is to show the existence of three non-trivial solutions for (P) without the subcritical growth condition for the term f. For this purpose, to overcome the lack of the Poincaré inequality, we need to introduce new functionals (see Section 2.1). By using these functionals, also under the Neumann boundary condition, we can prove the existence result as in the Dirichlet problems via super- and sub-solution, that is, the existence of a solution within the order interval between a sub-solution and a super-solution. Our main existence result provides two opposite constant sign solutions and a third one which is a sign changing (Theorem 1).

For example, because we do not impose the subcritical growth condition, we can deal with the following nonlinearities:

$$f(x,t) = \alpha_0 t_+^{p-1} - \beta_0 t_-^{p-1} - \alpha\eta(t)e^{t_+} + \beta\eta(t)e^{t_-}$$

for some positive constants α_0, β_0, α and β, where $t_\pm := \max\{\pm t, 0\}$ and η is a function such that $\eta(t) = 0$ for small $|t|$ and $\eta(t) = 1$ for large $|t|$. Moreover, we can apply our result to (P) with a jumping nonlinearity f as treated in [19].

In Section 2.2, to show our main result, we develop our approach via a super-solution and a sub-solution for the Neumann boundary condition. In Section 3, we present a result concerning local minimizers in $C^1(\overline{\Omega})$ and $W^{1,p}(\Omega)$ under the subcritical growth condition by using an argument originating in [17]. The paper also contains auxiliary results which are of independent interest such as the density of the space $C_n^1(\overline{\Omega})$ in $W^{1,p}(\Omega)$ (see Section 3), a strong maximum principle of Zhang's type, and a Moser iteration result for quasilinear elliptic equations depending on a parameter (see Appendix A).

1.1. Statements of main results

To state our existence results, we introduce several conditions for f which are not necessarily simultaneously assumed in our results. In the sequel, we set $F(x,u) := \int_0^u f(x,s)\,ds$.

- $(f+)$ There exists a $T^+ > 0$ such that $f(x,T^+) \leqslant 0$ for a.e. $x \in \Omega$.
- $(f-)$ There exists a $T^- < 0$ such that $f(x,T^-) \geqslant 0$ for a.e. $x \in \Omega$.

$(f0+)$ There exist a measurable subset Ω' of Ω and $\delta_0 > 0$ such that $\mu(\Omega') > 0$,

$$f(x,t) \geqslant 0 \quad \text{for every } 0 < t \leqslant \delta_0,\ \text{a.e. } x \in \Omega \quad \text{and}$$
$$f(x,t) > 0 \quad \text{for every } 0 < t \leqslant \delta_0,\ \text{a.e. } x \in \Omega',$$

where $\mu(X)$ denotes the Lebesgue measure of a measurable set X.

$(f0-)$ There exist a measurable subset Ω' of Ω and $\delta_1 > 0$ such that $\mu(\Omega') > 0$,

$$f(x,t) \leqslant 0 \quad \text{for every } 0 > t \geqslant -\delta_1,\ \text{a.e. } x \in \Omega \quad \text{and}$$
$$f(x,t) < 0 \quad \text{for every } 0 > t \geqslant -\delta_1,\ \text{a.e. } x \in \Omega'.$$

$(\widetilde{f0+})$ There exist $\lambda > 0$ and $\delta_2 > 0$ such that $f(x,t) \geqslant -\lambda t^{p-1}(1 + |\log t|)$ for every $0 < t \leqslant \delta_2$, a.e. $x \in \Omega$.

$(\widetilde{f0-})$ There exist $\lambda > 0$ and $\delta_3 > 0$ such that $f(x,t) \leqslant \lambda |t|^{p-1}(1 + |\log t|)$ for every $-\delta_3 \leqslant t < 0$, a.e. $x \in \Omega$.

$(F0)$ There exist $\delta_4 > 0$ and $(\alpha, \beta) \in \mathscr{C}$ such that

$$\frac{\alpha C_1 t^p}{p(p-1)} \leqslant F(x,t) \quad \text{for every } 0 \leqslant t \leqslant \delta_4,\ \text{a.e. } x \in \Omega,$$
$$\frac{\beta C_1 |t|^p}{p(p-1)} \leqslant F(x,t) \quad \text{for every } -\delta_4 \leqslant t \leqslant 0,\ \text{a.e. } x \in \Omega,$$

where $C_1 > 0$ is the constant in (A) and $\mathscr{C}$ denotes the first non-trivial curve defined by (1) (see Section 1.2 for the definition).

First, we state our main existence result:

Theorem 1. *If $(f+)$, $(f-)$, $(F0)$, $(f0+)$ and $(f0-)$ hold, then* (P) *has a positive solution, a negative solution and a sign-changing solution.*

Without a local sign condition on f at 0, we can obtain the following existence result.

Theorem 2. *If $(f+)$, $(f-)$, $(F0)$, $(\widetilde{f0+})$ and $(\widetilde{f0-})$ hold, then* (P) *has a positive solution, a negative solution and another non-trivial solution.*

Remark 3. We remark that there exists a positive solution $u_s \in C^1(\overline{\Omega})$ or a negative solution $u_l \in C^1(\overline{\Omega})$ of (P) such that $\delta_0 < \max_{\overline{\Omega}} u_s \leqslant T^+$ or $-\delta_1 > \min_{\overline{\Omega}} u_l \geqslant T^-$ if either $(f+)$ and $(f0+)$, or $(f-)$ and $(f0-)$ hold, respectively (see Propositions 19 and 20). Furthermore, we mention that $(f+)$ (resp. $(f-)$) can be replaced with the assumption that (P) has a positive super-solution (resp. a negative sub-solution) belonging to $W^{1,p}(\Omega) \cap L^\infty(\Omega)$.

As a byproduct of the proof of Theorem 1, we have also the following results in the case where f does not change sign either on toward $+\infty$ or toward $-\infty$.

Theorem 4. *Assume* $(f+)$, $(F0)$, $(f0+)$, $(f0-)$ *and the following* (i) *or* (ii):

(i) *there exists a constant* $C > 0$ *such that* $C \geqslant f(x,t) \geqslant -C(1+|t|^{p-1})$ *for every* $t \leqslant 0$ *and a.e.* $x \in \Omega$ *and* $\lim_{t\to-\infty} f(x,t) = -\infty$ *for a.e.* $x \in \Omega$;
(ii) *there exists a constant* $C > 0$ *such that* $C \geqslant f(x,t) \geqslant -C$ *for every* $t \leqslant 0$ *and a.e.* $x \in \Omega$ *and* $\limsup_{t\to-\infty} f(x,t) < 0$ *for a.e.* $x \in \Omega$.

Then, (P) *has a positive solution and at least one sign-changing solution.*

Theorem 5. *Assume* $(f-)$, $(F0)$, $(f0+)$, $(f0-)$ *and the following* (i) *or* (ii):

(i) *there exists a constant* $C > 0$ *such that* $-C \leqslant f(x,t) \leqslant C(1+|t|^{p-1})$ *for every* $t \geqslant 0$ *and a.e.* $x \in \Omega$ *and* $\lim_{t\to+\infty} f(x,t) = +\infty$ *for a.e.* $x \in \Omega$;
(ii) *there exists a constant* $C > 0$ *such that* $-C \leqslant f(x,t) \leqslant C$ *for every* $t \geqslant 0$ *and a.e.* $x \in \Omega$ *and* $\liminf_{t\to+\infty} f(x,t) > 0$ *for a.e.* $x \in \Omega$.

Then, (P) *has a negative solution and at least one sign-changing solution.*

1.2. The definition of the first non-trivial curve $\mathscr{C}$

In this subsection, we recall the result for the special case of $A(x,y) = |y|^{p-2}y$, that is, p-Laplacian problems (note that we can take $C_0 = C_1 = p-1$ in (A)). The construction of the curve $\mathscr{C}$ contained in the Fučík spectrum is carried out along the same lines as in [12], where the Dirichlet boundary condition is concerned: For $s \geqslant 0$, we define

$$J_s(u) := \int_\Omega |\nabla u|^p\,dx - s\int_\Omega u_+^p\,dx \quad \text{for } u \in W^{1,p}(\Omega), \qquad \tilde{J}_s := J_s|_S,$$
$$S := \left\{u \in W^{1,p}(\Omega);\ \int_\Omega |u|^p\,dx = 1\right\},$$
$$\Sigma := \bigl\{\gamma \in C\bigl([0,1],S\bigr);\ \gamma(0) = \psi_1,\ \gamma(1) = -\psi_1\bigr\},$$

where $u_\pm := \max\{\pm u, 0\}$ and $\psi_1 = 1/\mu(\Omega)^{1/p}$ (so $\|\psi_1\|_p = 1$). Here, $C([0,1],S)$ denotes the set of all continuous functions from $[0,1]$ to S with the topology induced by the $W^{1,p}(\Omega)$ norm. Finally, we set

$$c(s) := \inf_{\gamma\in\Sigma}\max_{t\in[0,1]} \tilde{J}_s\bigl(\gamma(t)\bigr).$$

It can be proved that $c(s)$ is a positive critical value of $\tilde{J}_s$ with $c(0) = \mu_2$, where μ_2 is the second eigenvalue of $-\Delta_p$ under the Neumann boundary condition. Moreover, we can see that $c(s)$ is continuous, nonincreasing in $s \geqslant 0$ and $c(s)+s$ is nondecreasing in $s \geqslant 0$.

Then, $\mathscr{C}$ is defined as follows:

$$\mathscr{C} := \bigl\{\bigl(c(s)+s, c(s)\bigr);\ s \geqslant 0\bigr\} \cup \bigl\{\bigl(c(s), c(s)+s\bigr);\ s \geqslant 0\bigr\}. \tag{1}$$

Remark 6. By using the Lagrange multiplier rule, a critical point $u \in S$ of $\tilde{J}_s$ with $\tilde{J}_s(u) = c(s)$ corresponds to a non-trivial solution belonging to $C^{1,\nu}(\overline{\Omega})$ ($\nu \in (0,1)$) of $-\Delta_p u = (c(s)+s)u_+^{p-1} - c(s)u_-^{p-1}$ in Ω and $\partial u/\partial n = 0$ on $\partial\Omega$. Moreover, it follows from [19, Proposition 2] that the solution u changes sign. Because the p-Laplace operator is $(p-1)$ homogeneous, if u is a solution of $-\Delta_p u = \alpha u_+^{p-1} - \beta u_-^{p-1}$ in Ω for some $\alpha, \beta \in \mathbb{R}$, then tu is also a solution of it for every $t \geqslant 0$.

2. Preliminaries

In what follows, the norm on $W^{1,p}(\Omega)$ is given by $\|u\|^p := \|\nabla u\|_p^p + \|u\|_p^p$, where $\|u\|_q$ denotes the norm of $L^q(\Omega)$ for $u \in L^q(\Omega)$ ($1 \leqslant q \leqslant \infty$). Setting $G(x,y) := \int_0^{|y|} a(x,t)t\,dt$, then we can easily see that

$$\nabla_y G(x,y) = A(x,y) \quad \text{and} \quad G(x,0) = 0$$

for every $x \in \overline{\Omega}$ (see [16] for details).

Remark 7. In [16], the following inequality is assumed with some $\eta \in L^1(\Omega)$:

$$pG(x,y) - A(x,y)y \geqslant \eta(x) \quad \text{for a.e. } x \in \Omega \text{ and every } y \in \mathbb{R}^N. \tag{2}$$

Relation (2) is used to prove only the boundedness of the Cerami sequence. Here, we do not need inequality (2). Moreover, it is easily seen that the following assertions hold under condition (A):

(i) for all $x \in \overline{\Omega}$, $A(x,y)$ is maximal monotone and strictly monotone in y;
(ii) $|A(x,y)| \leqslant \frac{C_1}{p-1}|y|^{p-1}$ for every $(x,y) \in \overline{\Omega} \times \mathbb{R}^N$;
(iii) $A(x,y)y \geqslant \frac{C_0}{p-1}|y|^p$ for every $(x,y) \in \overline{\Omega} \times \mathbb{R}^N$;
(iv) $G(x,y)$ is convex in y for all x and satisfies the following inequalities:

$$A(x,y)y \geqslant G(x,y) \geqslant \frac{C_0}{p(p-1)}|y|^p \quad \text{and} \quad G(x,y) \leqslant \frac{C_1}{p(p-1)}|y|^p \tag{3}$$

for every $(x,y) \in \overline{\Omega} \times \mathbb{R}^N$,

where C_0 and C_1 are the positive constants in (A).

Remark 8. We remark the following:

(i) If $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a solution of (P), then $u \in C^{1,\nu}(\overline{\Omega})$ for some $0 < \nu < 1$ and $\partial u/\partial n = 0$ on $\partial\Omega$;
(ii) Under assumption $(\widetilde{f0+})$, if $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a non-trivial solution of (P) such that $u \geqslant 0$, then u is a positive solution of (P) and $\min_{\overline{\Omega}} u > 0$ holds;
(iii) Under assumption $(\widetilde{f0-})$, if $v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a non-trivial solution of (P) satisfying $v \leqslant 0$, then $\max_{\overline{\Omega}} v < 0$ holds.

Proof. For readers' convenience, we give a sketch of the proof. (i): Let $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a solution of (P), namely, u satisfies the equality

$$\int_\Omega A(x, \nabla u)\nabla\varphi\, dx = \int_\Omega f(x, u)\varphi\, dx$$

for every $\varphi \in W^{1,p}(\Omega)$. Then, because of $u \in L^\infty(\Omega)$, we see that $u \in C^{1,\nu}(\overline{\Omega})$ $(0 < \nu < 1)$ by the regularity result in [14]. Furthermore, by [10, Theorem 3] (note $f(\cdot, u(\cdot)) \in L^\infty(\Omega)$), u satisfies the boundary condition

$$0 = \frac{\partial u}{\partial n_A} = A(\cdot, \nabla u)n = a\bigl(\cdot, |\nabla u|\bigr)\frac{\partial u}{\partial n} \quad \text{in } W^{-1/q,q}(\partial\Omega)$$

for every $1 < q < \infty$ (see [10] for the definition of $W^{-1/q,q}(\partial\Omega)$). Since $u \in C^{1,\nu}(\overline{\Omega})$ and $a(x,t) > 0$ for every $t \neq 0$, u satisfies the Neumann boundary condition, that is, $\frac{\partial u}{\partial n}(x) = 0$ for every $x \in \partial\Omega$.

(ii): By the boundedness of f on bounded sets, there exists a $d > 0$ such that $f(x,t) \geqslant -d \geqslant -dt^{p-1}/\delta_2^{p-1}$ for every $t \in [\delta_2, \delta_2 + \|u\|_\infty]$, a.e. $x \in \Omega$, where δ_2 is the positive constant in $(\widetilde{f0+})$. Define a function ξ by $\xi(t) := t^{p-1}(1 + |\log t|)$ if $t > 0$ and $\xi(t) = 0$ if $t \leqslant 0$. Hence, we have

$$-\operatorname{div} A(x, \nabla u) + \max\bigl\{\lambda, d/\delta_2^{p-1}\bigr\}\xi(u) = f(x,u) + \max\bigl\{\lambda, d/\delta_2^{p-1}\bigr\}\xi(u) \geqslant 0 \quad \text{in } \Omega,$$

where λ is the positive constant in $(\widetilde{f0+})$. By noting that $u \in C^{1,\nu}(\overline{\Omega})$ $(0 < \nu < 1)$ by (i), we have $u(x) > 0$ for every $x \in \Omega$ by Theorem B in Appendix A. Due to the strong maximum principle (see Theorem A in Appendix A), we easily see that $u(x) > 0$ for every $x \in \partial\Omega$ (note $\partial u/\partial n = 0$ on $\partial\Omega$ by (i)). This yields $\min_{\overline{\Omega}} u > 0$ because of $u \in C^{1,\nu}(\overline{\Omega})$ by (i).

(iii): Similarly, because $-\operatorname{div} A(x, \nabla v_-) + \lambda'\xi(v_-) \geqslant 0$ in Ω holds for some positive constant λ' (note that A is odd in y), we have $\min_{\overline{\Omega}} v_- > 0$ by the same reasoning as for (ii). Hence, $\max_{\overline{\Omega}} v < 0$ holds. □

It is well known that the Palais–Smale condition, whose formulation we now recall, plays an important role in the mountain pass argument.

Definition 9. A C^1 functional J on a Banach space X is said to satisfy the Palais–Smale condition at $c \in \mathbb{R}$ if every sequence $\{u_m\} \subset X$ satisfying

$$J(u_m) \to c \quad \text{and} \quad \bigl\|J'(u_m)\bigr\|_{X^*} \to 0 \quad \text{as } n \to \infty$$

has a convergent subsequence. We say that J satisfies the Palais–Smale condition if J satisfies the Palais–Smale condition at any $c \in \mathbb{R}$.

The following result is important for the proof of the Palais–Smale condition for the functionals related to our problem. Notice that we do not need (2) to prove it.

Proposition 10. *(See [16, Proposition 1].) Let* $V : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ *be the map defined by*

$$\langle V(u), v\rangle = \int_\Omega A(x, \nabla u)\nabla v\, dx$$

for $u, v \in W^{1,p}(\Omega)$*. Then,* V *is maximal monotone, strictly monotone and has the* $(S)_+$ *property, that is, any sequence* $\{u_m\}$ *weakly convergent to* u *strongly converges to* u *provided* $\limsup_{m\to\infty}\langle V(u_m), u_m - u\rangle \leqslant 0$.

Let us introduce the subcritical growth condition (g):

(g) there exist $C > 0$ and $1 \leqslant r < p^*$ such that

$$\left|f(x,t)\right| \leqslant C\left(1 + |t|^{r-1}\right) \quad \text{for every } t \in \mathbb{R}, \text{ a.e. } x \in \Omega,$$

where $p^* = pN/(N-p)$ if $N > p$ and $p^* = \infty$ if $N \leqslant p$.

It is well known that the following functional I on $W^{1,p}(\Omega)$ is of class C^1 under (g):

$$I(v) := \int_\Omega G(x, \nabla v)\, dx - \int_\Omega F(x, v)\, dx \tag{4}$$

for $v \in W^{1,p}(\Omega)$. Furthermore, by the $(S)_+$ property of the operator V and the compactness of the inclusion of $W^{1,p}(\Omega)$ into $L^r(\Omega)$, we can prove that I satisfies the bounded Palais–Smale condition, that is, any bounded Palais–Smale sequence for I has a convergent subsequence.

2.1. Modifications to the functional via truncation

It is one of the features of our paper that the nonlinear term $f(x,t)$ is not assumed to satisfy suitable growth conditions as $|t| \to \infty$. Therefore, problem (P) cannot be directly transformed to the problem of finding a critical point of some functional on $W^{1,p}(\Omega)$. So, we use the truncations $f_{[\underline{u},\overline{u}]}$, $f_{[-\infty,\overline{u}]}$ and $f_{[\underline{u},+\infty]}$ of f defined by two $L^\infty(\Omega)$ functions $\underline{u}$ and $\overline{u}$ with $\underline{u} \leqslant \overline{u}$ (a.e. in Ω) as follows:

$$f_{[\underline{u},\overline{u}]}(x,t) := \begin{cases} f(x, \underline{u}(x)) & \text{if } t \leqslant \underline{u}(x), \\ f(x,t) & \text{if } \underline{u}(x) < t < \overline{u}(x), \\ f(x, \overline{u}(x)) & \text{if } t \geqslant \overline{u}(x), \end{cases}$$

$$f_{[-\infty,\overline{u}]}(x,t) = \begin{cases} f(x,t) & \text{if } t < \overline{u}(x), \\ f(x, \overline{u}(x)) & \text{if } t \geqslant \overline{u}(x), \end{cases} \qquad f_{[\underline{u},+\infty]}(x,t) = \begin{cases} f(x, \underline{u}(x)) & \text{if } t \leqslant \underline{u}(x), \\ f(x,t) & \text{if } t > \underline{u}(x). \end{cases}$$

Set $F_{[z,w]}(x,u) := \int_0^u f_{[z,w]}(x,t)\, dt$ for $[z,w] = [\underline{u},\overline{u}]$, $[-\infty,\overline{u}]$ or $[\underline{u},+\infty]$. Then, we define a C^1 functional $I_{[\underline{u},\overline{u}]}$ on $W^{1,p}(\Omega)$ by

$$I_{[\underline{u},\overline{u}]}(u) := \int_\Omega G(x, \nabla u)\, dx - \int_\Omega F_{[\underline{u},\overline{u}]}(x,u)\, dx + \left\|(u-\overline{u})_+\right\|_p^p + \left\|(u-\underline{u})_-\right\|_p^p \tag{5}$$

for $u \in W^{1,p}(\Omega)$. Note that $f_{[\underline{u},\overline{u}]}(x,t) + p(t-\overline{u}(x))_+^{p-1} - p(t-\underline{u}(x))_-^{p-1}$ satisfies the subcritical growth condition (g). Similarly, we introduce two functionals $I_{[-\infty,\overline{u}]}$ and $I_{[\underline{u},+\infty]}$ on $W^{1,p}(\Omega)$ defined by

$$I_{[-\infty,\overline{u}]}(u) := \int_\Omega G(x,\nabla u)\,dx - \int_\Omega F_{[-\infty,\overline{u}]}(x,u)\,dx + \left\|(u-\overline{u})_+\right\|_p^p, \tag{6}$$

$$I_{[\underline{u},+\infty]}(u) := \int_\Omega G(x,\nabla u)\,dx - \int_\Omega F_{[\underline{u},+\infty]}(x,u)\,dx + \left\|(u-\underline{u})_-\right\|_p^p \tag{7}$$

for $u \in W^{1,p}(\Omega)$.

Let us show the following elementary result which implies the existence of a global minimizer (cf. [15, Theorem 1.1.]). It is well known that it guarantees also the Palais–Smale condition of $I_{[\underline{u},\overline{u}]}$.

Lemma 11. *Let $\underline{u},\overline{u} \in L^\infty(\Omega)$ satisfy $\underline{u} \leqslant \overline{u}$ (a.e. on Ω). Then, $I_{[\underline{u},\overline{u}]}$ defined by* (5) *is weakly lower semi-continuous, bounded from below and coercive.*

Proof. Because f is bounded on $\Omega \times [-\|\underline{u}\|_\infty, \|\overline{u}\|_\infty]$, there exists a positive constant d such that $|f_{[\underline{u},\overline{u}]}(x,t)| \leqslant d$ for every $t \in \mathbb{R}$, a.e. $x \in \Omega$. Thus, we have

$$I_{[\underline{u},\overline{u}]}(u) \geqslant \frac{C_0}{p(p-1)}\|\nabla u\|_p^p - d\|u\|_1 + \left\|(u-\overline{u})_+\right\|_p^p + \left\|(u-\underline{u})_-\right\|_p^p \tag{8}$$

for every $u \in W^{1,p}(\Omega)$ by (3). It is easily shown that for every $\varepsilon > 0$ there exists a positive constant C_ε such that

$$\left||a+b|^p - |a|^p\right| \leqslant \varepsilon|a|^p + C_\varepsilon|b|^p \quad \text{for every } a,b \in \mathbb{R}. \tag{9}$$

We fix ε such that $0 < \varepsilon < 1$. Then, we obtain the following inequality:

$$\begin{aligned}\int_\Omega (u-\overline{u})_+^p\,dx &= \int_\Omega u_+^p\,dx - \int_{u<\overline{u}_+} u_+^p\,dx + \int_{u<\overline{u}_+} (u-\overline{u})_+^p\,dx \\ &\quad + \int_{u\geqslant\overline{u}_+} \left((u-\overline{u})^p - u^p\right)dx \\ &\geqslant \|u_+\|_p^p - \int_\Omega \overline{u}_+^p\,dx - \int_{u\geqslant\overline{u}_+} \left(\varepsilon u^p + C_\varepsilon|\overline{u}|^p\right)dx \\ &\geqslant (1-\varepsilon)\|u_+\|_p^p - (1+C_\varepsilon)\|\overline{u}\|_p^p \end{aligned} \tag{10}$$

by (9). Similarly, we have

$$\int_\Omega (u-\underline{u})_-^p\,dx \geqslant (1-\varepsilon)\|u_-\|_p^p - (1+C_\varepsilon)\|\underline{u}\|_p^p. \tag{11}$$

Therefore, by (8), (10) and (11), the functional $I_{[\underline{u},\overline{u}]}$ satisfies

$$I_{[\underline{u},\overline{u}]}(u) \geqslant \min\left\{\frac{C_0}{p(p-1)}, (1-\varepsilon)\right\}\|u\|^p - d'\|u\| - (1+C_\varepsilon)\|\overline{u}\|_p^p - (1+C_\varepsilon)\|\underline{u}\|_p^p$$

for every $u \in W^{1,p}(\Omega)$, where $d' > 0$ is a constant independent of u. This implies that $I_{[\underline{u},\overline{u}]}$ is bounded from below and coercive because of $p > 1$. Note that $\Phi(u) := \int_\Omega G(x, \nabla u)\,dx$ is weakly lower semi-continuous (w.l.s.c.) on $W^{1,p}(\Omega)$ because Φ is convex and continuous on $W^{1,p}(\Omega)$ (see [15, Theorem 1.2.]). Thus, $I_{[\underline{u},\overline{u}]}$ is also w.l.s.c. on $W^{1,p}(\Omega)$ since the inclusion of $W^{1,p}(\Omega)$ into $L^p(\Omega)$ is compact (note the boundedness of $f_{[\underline{u},\overline{u}]}$). $\square$

For the proofs of Theorems 4 and 5, we state the following lemma concerning the Palais–Smale condition.

Lemma 12. *Let $\overline{u} \in L^\infty(\Omega)$ (resp. $\underline{u} \in L^\infty(\Omega)$). Assume* (i) *or* (ii) *as in Theorem* 4 *(resp. Theorem* 5*). Then, $I_{[-\infty,\overline{u}]}$ (resp. $I_{[\underline{u},+\infty]}$) satisfies the Palais–Smale condition, where $I_{[-\infty,\overline{u}]}$ and $I_{[\underline{u},+\infty]}$ are the functionals defined by* (6) *and* (7)*, respectively.*

Proof. First, we note that $I_{[-\infty,\overline{u}]}$ and $I_{[\underline{u},+\infty]}$ are C^1 functionals on $W^{1,p}(\Omega)$ because $f_{[-\infty,\overline{u}]}$ and $f_{[\underline{u},+\infty]}$ satisfy the subcritical growth condition (g) under our assumptions. We treat only $I_{[-\infty,\overline{u}]}$ since $I_{[\underline{u},+\infty]}$ can be handled similarly. Let $\{u_m\}$ be a Palais–Smale sequence for $I_{[-\infty,\overline{u}]}$, namely, $I'_{[-\infty,\overline{u}]}(u_m) \to 0$ in $W^{1,p}(\Omega)^*$ and $I_{[-\infty,\overline{u}]}(u_m) \to c$ as $m \to \infty$ for some $c \in \mathbb{R}$. Because the operator V defined in Proposition 10 satisfies the $(S)_+$ property and $W^{1,p}(\Omega)$ is embedded compactly into $L^p(\Omega)$, it is sufficient to show the boundedness of $\|u_m\|$.

Case of assumption (i): By taking u_{m+} as test function, we have

$$\begin{aligned}
o(1)\|u_{m+}\| &= \left\langle I'_{[-\infty,\overline{u}]}(u_m), u_{m+}\right\rangle \\
&= \int_\Omega A(x, \nabla u_{m+})\nabla u_{m+}\,dx - \int_\Omega f_{[-\infty,\overline{u}]}(x, u_{m+})u_{m+}\,dx \\
&\quad + p\int_\Omega (u_{m+} - \overline{u})_+^{p-1} u_{m+}\,dx \\
&\geqslant \frac{C_0}{p-1}\|\nabla u_{m+}\|_p^p - d_1\|u_{m+}\|_1 + \|u_{m+}\|_p^p - d_2\|\overline{u}_+\|_p^p
\end{aligned}$$

by Remark 7(iii), the boundedness of f on $\Omega \times [-\|\overline{u}\|_\infty, \|\overline{u}\|_\infty]$, (10) (with $\varepsilon = 1 - 1/p$) and the following inequality:

$$\int_\Omega (u_{m+} - \overline{u})_+^{p-1} u_{m+}\,dx \geqslant \int_{u_{m+} \geqslant \overline{u}_+} (u_{m+} - \overline{u}_+)_+^p\,dx = \int_\Omega (u_{m+} - \overline{u}_+)_+^p\,dx,$$

where d_1 and d_2 are positive constants independent of m. This shows the boundedness of $\|u_{m+}\|$ (note $p > 1$). On the other hand, by taking $-u_{m-}$ as test function, we have

$$\begin{aligned}
o(1)\|u_{m-}\| &= \left\langle I'_{[-\infty,\bar u]}(u_m), -u_{m-}\right\rangle \\
&= \int_\Omega A(x,-\nabla u_{m-})(-\nabla u_{m-})\,dx + \int_\Omega f_{[-\infty,\bar u]}(x,-u_{m-})u_{m-}\,dx \\
&\quad - p\int_{\bar u\leqslant u_m\leqslant 0}(u_m-\bar u)_+^{p-1}u_{m-}\,dx \\
&\geqslant \frac{C_0}{p-1}\|\nabla u_{m-}\|_p^p - d_3\|u_{m-}\|_p^p - d_3\|u_{m-}\|_1 - p\|\bar u_-\|_p^p \qquad (12)
\end{aligned}$$

since $f_{[-\infty,\bar u]}$ satisfies an inequality as in (i) on $\Omega\times(-\infty,\|\bar u\|_\infty]$ for some positive constant d_3. To obtain the boundedness of $\|u_m\|$, by combining the boundedness of $\|u_{m+}\|$ and (12), it suffices to show that $\|u_{m-}\|_p$ is bounded. Assume $\|u_{m-}\|_p\to\infty$ by choosing a subsequence. Set $v_m := u_m/\|u_{m-}\|_p$. Then, $\|v_{m-}\|_p = 1$ and $\|v_{m+}\|_p\leqslant\|u_{m+}\|_p/\|u_{m-}\|_p\to 0$ as $m\to\infty$ (note that $\|u_{m+}\|$ is bounded). Because $\|v_m\|$ is bounded due to (12), we may also suppose, by taking a subsequence, that there exists a non-positive function $v_0\in W^{1,p}(\Omega)$ such that v_m converges to v_0 weakly in $W^{1,p}(\Omega)$, strongly in $L^p(\Omega)$, a.e. $x\in\Omega$ and $\|v_0\|_p=1$. Hence, $u_m(x)\to-\infty$ for a.e. $x\in\Omega' := \{x\in\Omega;\ v_0(x)\neq 0\}$. Then, by taking $\varphi\equiv 1$ as test function in $I'_{[-\infty,\bar u]}(u_m)\to 0$ in $W^{1,p}(\Omega)^*$, we obtain

$$\begin{aligned}
o(1) &= -\int_\Omega f_{[-\infty,\bar u]}(x,-u_{m-})\,dx - \int_\Omega f_{[-\infty,\bar u]}(x,u_{m+})\,dx + p\int_\Omega (u_m-\bar u)_+^{p-1}\,dx \\
&\geqslant -\int_{\Omega'} f_{[-\infty,\bar u]}(x,-u_{m-})\,dx - d_4\left|\Omega\setminus\Omega'\right| - d_5
\end{aligned}$$

for some positive constants d_4 and d_5 independent of m because $f_{[-\infty,\bar u]}$ is bounded from above on $\Omega\times\mathbb{R}$. Therefore, by passing to the limit inferior in the inequality above, we achieve a contradiction through the Fatou lemma applied to $-f_{[-\infty,\bar u]}(x,-u_{m-})$ (note $\mu(\Omega')>0$).

Case of assumption (ii): Because $f_{[-\infty,\bar u]}$ is bounded in this case, we see that $\|u_{m+}\|$ is bounded and $\|\nabla u_{m-}\|_p^p\leqslant C(\|u_{m-}\|_p+1)$ for some $C>0$ independent of m by an argument similar to assumption (i) (we use $o(1)\|u_{m-}\|\geqslant C_0\|\nabla u_{m-}\|_p^p/(p-1) - d_3\|u_{m-}\|_1 - p\|\bar u_-\|_p^p$ instead of (12)). So, since our purpose here is to show the boundedness of $\|u_{m-}\|_p$, by way of contradiction, we may assume that $\|u_{m-}\|_p\to+\infty$ along a subsequence. Set $v_m := u_m/\|u_{m-}\|_p$, and then $\|\nabla v_{m-}\|_p^p\leqslant C(\|u_{m-}\|_p+1)/\|u_{m-}\|_p^p\to 0$ and $\|v_{m+}\|\to 0$ as $m\to\infty$ (note that $\|u_{m+}\|$ is bounded and $p>1$). Hence, we may suppose that v_m converges to $-\psi_1$ strongly in $W^{1,p}(\Omega)$ and a.e. $x\in\Omega$, where $\psi_1=\mu(\Omega)^{-1/p}$ (note $\|\psi_1\|_p=1$). Therefore, $u_m(x)\to-\infty$ for a.e. $x\in\Omega$. Due to Fatou's lemma, we have a contradiction by passing to the limit inferior in the following inequality:

$$o(1) = -\int_\Omega f_{[-\infty,\bar u]}(x,u_m)\,dx + p\int_\Omega (u_m-\bar u)_+^{p-1}\,dx \geqslant -\int_\Omega f_{[-\infty,\bar u]}(x,u_m)\,dx. \qquad \square$$

2.2. Super-solutions and sub-solutions

Definition 13. We say that $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a super-solution (resp. sub-solution) of (P) if u satisfies

$$\int_\Omega A(x, \nabla u)\nabla\varphi\, dx \geqslant \int_\Omega f(x,u)\varphi\, dx \quad \left(\text{resp. } \int_\Omega A(x, \nabla u)\nabla\varphi\, dx \leqslant \int_\Omega f(x,u)\varphi\, dx\right)$$

for every $\varphi \in W^{1,p}(\Omega)$ with $\varphi \geqslant 0$.

Throughout this subsection, we assume that $\overline{u} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a super-solution of (P) and $\underline{u} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a sub-solution of (P) with $\underline{u} \leqslant \overline{u}$ (a.e. in Ω).

The following result plays an important role to show the existence of a solution for (P).

Lemma 14. *If $u \in W^{1,p}(\Omega)$ is a critical point of $I_{[\underline{u},\overline{u}]}$, then $u \in C^{1,\nu}(\overline{\Omega})$ (for some $\nu \in (0,1)$) and u is a solution of* (P) *with $\underline{u} \leqslant u \leqslant \overline{u}$.*

Proof. Let $u \in W^{1,p}(\Omega)$ be a critical point of $I_{[\underline{u},\overline{u}]}$, so u satisfies

$$\int_\Omega A(x, \nabla u)\nabla\varphi\, dx = \int_\Omega f_{[\underline{u},\overline{u}]}(x,u)\varphi\, dx - p\int_\Omega (u-\overline{u})_+^{p-1}\varphi\, dx + p\int_\Omega (u-\underline{u})_-^{p-1}\varphi\, dx$$

for every $\varphi \in W^{1,p}(\Omega)$. Because $\overline{u}$ and $\underline{u}$ are a super-solution and a sub-solution of (P) respectively, we have $\langle I'_{[\underline{u},\overline{u}]}(\overline{u}), \varphi\rangle \geqslant 0$ and $\langle I'_{[\underline{u},\overline{u}]}(\underline{u}), \varphi\rangle \leqslant 0$ for every $\varphi \in W^{1,p}(\Omega)$ with $\varphi \geqslant 0$ (note $f_{[\underline{u},\overline{u}]}(x, \underline{u}(x)) = f(x, \underline{u}(x))$ and $f_{[\underline{u},\overline{u}]}(x, \overline{u}(x)) = f(x, \overline{u}(x))$). Therefore, we obtain $\|(u-\overline{u})_+\|_p = 0$, and so $u(x) \leqslant \overline{u}(x)$ for a.e. $x \in \Omega$ by the following inequality:

$$\begin{aligned}
0 &\geqslant \big\langle I'_{[\underline{u},\overline{u}]}(u) - I'_{[\underline{u},\overline{u}]}(\overline{u}), (u-\overline{u})_+\big\rangle \\
&= \int_\Omega \big(A(x,\nabla u) - A(x, \nabla\overline{u})\big)\nabla(u-\overline{u})_+\, dx + p\big\|(u-\overline{u})_+\big\|_p^p \\
&\quad - \int_\Omega \big(f_{[\underline{u},\overline{u}]}(x,u) - f_{[\underline{u},\overline{u}]}(x,\overline{u})\big)(u-\overline{u})_+\, dx - p\int_\Omega (u-\underline{u})_-^{p-1}(u-\overline{u})_+\, dx \\
&= \int_{u\geqslant\overline{u}} \big(A(x,\nabla u) - A(x,\nabla\overline{u})\big)(\nabla u - \nabla\overline{u})\, dx + p\big\|(u-\overline{u})_+\big\|_p^p \geqslant 0
\end{aligned}$$

(note $\overline{u} \geqslant \underline{u}$ and also that the map A is strictly monotone in the second variable). Similarly, by considering $0 \geqslant \langle I'_{[\underline{u},\overline{u}]}(u) - I'_{[\underline{u},\overline{u}]}(\underline{u}), -(u-\underline{u})_-\rangle$, we see that $u \geqslant \underline{u}$ in Ω. As a result, $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a solution of (P) satisfying $\underline{u} \leqslant u \leqslant \overline{u}$ because of $f_{[\underline{u},\overline{u}]}(x,u(x)) = f(x,u(x))$. Moreover, by the regularity result (cf. [14]), $u \in C^{1,\nu}(\overline{\Omega})$ (for some $0 < \nu < 1$) holds (note $u \in L^\infty(\Omega)$). $\square$

Lemma 15. *Assume that $f_{[\underline{u},+\infty]}$ (resp. $f_{[-\infty,\overline{u}]}$) satisfies the subcritical growth condition (g). If $u \in W^{1,p}(\Omega)$ is a critical point of $I_{[\underline{u},+\infty]}$ (resp. $I_{[-\infty,\overline{u}]}$), then u is a solution belonging to $C^{1,\nu}(\overline{\Omega})$ (for some $0 < \nu < 1$) of* (P) *with $u \geqslant \underline{u}$ (resp. $u \leqslant \overline{u}$), where $I_{[\underline{u},+\infty]}$ and $I_{[-\infty,\overline{u}]}$ are the functionals defined by* (7) *and* (6)*, respectively.*

Proof. We treat only the case of $I_{[\underline{u},+\infty]}$ since the other case can be handled similarly. First, note that $I_{[\underline{u},+\infty]}$ is of C^1 class because $f_{[\underline{u},+\infty]}$ satisfies the subcritical growth condition under (g). Moreover, by the Moser iteration process (see Theorem C in Appendix A with $u_0 = 0$ and $\mu = 0$), any critical point of $I_{[\underline{u},+\infty]}$ belongs to $L^\infty(\Omega)$.

Let u be a critical point of $I_{[\underline{u},+\infty]}$. By the same argument as in the proof of Lemma 14, we obtain

$$\begin{aligned} 0 &\geqslant \big\langle I'_{[\underline{u},+\infty]}(u) - I'_{[\underline{u},+\infty]}(\underline{u}), -(u-\underline{u})_- \big\rangle \\ &\geqslant \int\limits_{u \leqslant \underline{u}} \big(A(x,\nabla u) - A(x,\nabla \underline{u})\big)(\nabla u - \nabla \underline{u})\,dx + p\big\|(u-\underline{u})_-\big\|_p^p \geqslant 0. \end{aligned}$$

This implies $u(x) \geqslant \underline{u}(x)$ for a.e. $x \in \Omega$. Thus, as for Lemma 14, we see that u is a solution of (P) belonging to $C^{1,\nu}(\overline{\Omega})$ for some $0 < \nu < 1$ (note $u \in L^\infty(\Omega)$). □

We state the following result via a super-solution and a sub-solution. It is well known for *the Dirichlet boundary condition* (e.g. [8, Chapter 3]).

Proposition 16. *The following assertions hold*:

(i) *If $\overline{u}_1, \overline{u}_2 \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ are super-solutions of* (P)*, then* $\min\{\overline{u}_1, \overline{u}_2\}$ *is also a super-solution of* (P);
(ii) *If $\underline{u}_1, \underline{u}_2 \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ are sub-solutions of* (P)*, then* $\max\{\underline{u}_1, \underline{u}_2\}$ *is also a sub-solution of* (P);
(iii) *If $\overline{u} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a super-solution of* (P) *and $\underline{u} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a sub-solution of* (P) *with $\underline{u} \leqslant \overline{u}$ a.e. in Ω, then there exists a solution $u \in C^{1,\nu}(\overline{\Omega})$ (for some $0 < \nu < 1$) of* (P) *within the order interval $[\underline{u}, \overline{u}]$. Moreover, there exist a smallest solution and a largest solution of* (P) *in $[\underline{u}, \overline{u}]$.*

Proof. By the same argument as in [1, Lemma 1] (which treats the p-Laplace equation under the Neumann boundary condition), we can establish the assertions (i) and (ii). So, we prove only (iii) (note that the subsequent argument is patterned from [8, Theorem 3.11]).

Let $\overline{u} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a super-solution of (P) and let $\underline{u} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a sub-solution of (P) with $\underline{u} \leqslant \overline{u}$ a.e. in Ω. It is well known that the properties of $I_{[\underline{u},\overline{u}]}$ stated in Lemma 11 imply the existence of a global minimizer u of $I_{[\underline{u},\overline{u}]}$ (cf. [15, Theorem 1.1.]). Moreover, it follows from Lemma 14 that $u \in C^{1,\nu}(\overline{\Omega})$ (for some $0 < \nu < 1$) is a solution of (P) with $\underline{u} \leqslant u \leqslant \overline{u}$. Thus, the first part of our conclusion holds.

Finally, we shall only show the existence of the smallest solution in $[\underline{u}, \overline{u}]$ because we can consider the existence of the largest solution similarly. Set

$$\mathscr{S} := \mathscr{S}\big([\underline{u}, \overline{u}]\big) := \big\{u \in W^{1,p}(\Omega);\ u \text{ is a solution of (P) with } u \in [\underline{u}, \overline{u}]\big\} \subset C^1(\overline{\Omega}),$$

where the last inclusion is obtained by the regularity result. We claim that $\mathscr{S}$ is compact in $W^{1,p}(\Omega)$. Indeed, let $\{v_m\}$ be a sequence contained in $\mathscr{S}$. Then, it is clear that $\|v_m\|_\infty \leqslant \max\{\|\underline{u}\|_\infty, \|\overline{u}\|_\infty\}$, and so $\{v_m\}$ is bounded in $L^p(\Omega)$. Moreover, we can get the boundedness of $\{v_m\}$ in $W^{1,p}(\Omega)$ by the following inequality

$$\frac{C_0}{p-1}\|\nabla v_m\|_p^p \leqslant \int_\Omega A(x, \nabla v_m)\nabla v_m\,dx = \int_\Omega f(x, v_m)v_m\,dx \leqslant d\|v_m\|_1,$$

where we use Remark 7(iii) in the first inequality and d is a positive constant determined only by the boundedness of f on $\Omega \times [-\|\underline{u}\|_\infty, \|\overline{u}\|_\infty]$. Therefore, by taking a subsequence, there exists a $v_0 \in W^{1,p}(\Omega)$ such that v_m converges to v_0 weakly in $W^{1,p}(\Omega)$, strongly in $L^p(\Omega)$ and a.e. $x \in \Omega$. Since v_m is a solution of (P),

$$\int_\Omega A(x, \nabla v_m)(\nabla v_m - \nabla v_0)\,dx = \int_\Omega f(x, v_m)(v_m - v_0)\,dx \tag{13}$$

holds for every $m \in \mathbb{N}$. Letting $m \to \infty$ in (13), we see that

$$\lim_{m\to\infty}\langle V(v_m), v_m - v_0\rangle = \lim_{m\to\infty}\int_\Omega A(x, \nabla v_m)(\nabla v_m - \nabla v_0)\,dx = 0$$

by Lebesgue's dominated convergence theorem, where V is the operator defined in Proposition 10. It follows from Proposition 10 that v_m is strongly convergent to v_0 in $W^{1,p}(\Omega)$. Therefore, v_0 is a solution of (P) by passing to the limit with respect to m in the equality $\int_\Omega A(x, \nabla v_m)\nabla\varphi\,dx = \int_\Omega f(x, v_m)\varphi\,dx$ for every $\varphi \in W^{1,p}(\Omega)$ (note $\underline{u} \leqslant v_m \leqslant \overline{u}$). Therefore, $v_0 \in \mathscr{S}$ and so, our claim is shown.

Because $W^{1,p}(\Omega)$ is separable, we may assume that there exists a countable set $Z := \{z_m \in \mathscr{S};\ m \in \mathbb{N}\}$ such that $\overline{Z} = \mathscr{S}$, where $\overline{Z}$ denotes the closure of Z in $W^{1,p}(\Omega)$ topology. We shall construct by induction a decreasing sequence $\{u_m\}_m$ (see (14)) contained in $\mathscr{S}$. Set $u_1 := z_1$. Suppose that $u_i \in \mathscr{S}$ $(1 \leqslant i \leqslant m)$ is constructed such that $u_1 \geqslant u_2 \geqslant \cdots \geqslant u_m$. Then, the first property in (iii) guarantees the existence of $u_{m+1} \in \mathscr{S}$ satisfying

$$\underline{u} \leqslant u_{m+1} \leqslant \min\{z_m, u_m\} \leqslant u_m \leqslant \overline{u} \tag{14}$$

since $\min\{z_m, u_m\}$ is a super-solution of (P) by (i). Therefore, we can construct a decreasing sequence $\{u_m\}_m$ in $\mathscr{S}$ such that $\underline{u} \leqslant u_m \leqslant z_k$ $(1 \leqslant k \leqslant m-1)$.

Here, because $\{u_m\}$ is a decreasing sequence in $L^p(\Omega)$ and $u_m \geqslant \underline{u}$, there exists $\inf_k u_k \in L^p(\Omega)$ such that $u_m \to \inf_k u_k$ in $L^p(\Omega)$. On the other hand, by the compactness of $\mathscr{S}$ in $W^{1,p}(\Omega)$, there exists a $u_0 \in W^{1,p}(\Omega)$ such that $u_m \to u_0$ in $W^{1,p}(\Omega)$, and hence $u_0 = \inf_k u_k \in W^{1,p}(\Omega)$ by the uniqueness of the limit. Note that $u_0 \in \mathscr{S}$ by the compactness of $\mathscr{S}$, and so u_0 is a solution of (P) within $[\underline{u}, \overline{u}]$. Because $u_0 = \inf_k u_k \leqslant u_{m+1} \leqslant z_m \leqslant \overline{u}$ holds for every $m \in \mathbb{N}$, we have $z_m \in [u_0, \overline{u}]$ for every $m \in \mathbb{N}$, whence $Z \subset [u_0, \overline{u}]$. Furthermore, by taking the closure of Z in $W^{1,p}(\Omega)$, we obtain $\mathscr{S} = \overline{Z} \subset \overline{[u_0, \overline{u}]} = [u_0, \overline{u}]$. This shows the smallness of u_0 in $[\underline{u}, \overline{u}]$ (note $u_0 \in \mathscr{S}$). $\square$

Lemma 17. *Assume* $(f0+)$. *Let* $v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *be a positive solution of* (P). *Then,* $\|v\|_\infty > \delta_0$ *holds, where* δ_0 *denotes the positive constant in* $(f0+)$.

Proof. Let $v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a positive solution of (P). Arguing by contradiction, we assume that $0 < \|v\|_\infty \leqslant \delta_0$. By Remark 8, we see that $v \in C^{1,\nu}(\overline{\Omega})$ (for some $0 < \nu < 1$) and it satisfies $\min_{\overline{\Omega}} v > 0$. Then, by taking $\varphi \equiv 1$ as test function, we have $\int_\Omega f(x, v)\,dx = 0$, and hence $f(x, v(x)) = 0$ for a.e. $x \in \Omega$ due to $(f0+)$ and $\|v\|_\infty \leqslant \delta_0$. This contradicts $f(x, v(x)) > 0$ for a.e. $x \in \Omega'$. $\square$

Since the same argument applies to a negative solution, we omit the proof.

Lemma 18. *Assume* $(f0-)$. *Let* $v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *be a negative solution of* (P). *Then,* $\|v\|_\infty > \delta_1$ *holds, where* δ_1 *denotes the positive constant in* $(f0-)$.

We show the existence of a positive solution.

Proposition 19. *Assume* $(f0+)$. *Let* $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *be a super-solution of* (P) *with* $\operatorname{ess\,inf}_\Omega u > 0$. *Then* (P) *has a smallest positive solution* $u_s \in C^{1,\nu}(\overline{\Omega})$ *(for some* $0 < \nu < 1$*) with* $u_s \leqslant u$ *in the sense that* $u_s \leqslant v$ *for every positive solution* $v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *of* (P).

Proof. Since $\operatorname{ess\,inf}_\Omega u > 0$ by assumption, we can take a sequence $\{\varepsilon_m\}$ such that $0 < \varepsilon_m < \min\{\delta_0, \operatorname{ess\,inf}_\Omega u\}$ and $\varepsilon_m \downarrow 0$ as $m \to \infty$. Then, since $-\operatorname{div} A(x, \nabla\varepsilon_m) = 0 \leqslant f(x, \varepsilon_m)$ in Ω by $(f0+)$, we see that the constant function ε_m is a sub-solution of (P) satisfying $\varepsilon_m < u(x)$ for a.e. $x \in \Omega$. Therefore, it follows from Proposition 16(iii) that for every m there exists the smallest solution $u_m \in [\varepsilon_m, u]$ of (P) within $[\varepsilon_m, u]$. We notice $u_{m+1} \leqslant u_m$ because of $u_m \in [\varepsilon_{m+1}, u]$ and the minimality of u_{m+1} in $[\varepsilon_{m+1}, u]$. Therefore, by the compactness of $\mathscr{S}([0, u])$ in $W^{1,p}(\Omega)$ (see the proof of Proposition 16 for the definition of $\mathscr{S}([0, u])$), there exists a solution $u_s \in W^{1,p}(\Omega)$ of (P) within $[0, u]$ such that $u_m \to u_s$ in $W^{1,p}(\Omega)$ and a.e. $x \in \Omega$.

Now, we shall prove that $u_s \neq 0$. We assume, by contradiction that $u_s = 0$. Here, we note that $u_m, u_s \in C^{1,\nu}(\overline{\Omega})$ for some $\nu \in (0, 1)$ and $\|u_m\|_{C^{1,\nu}(\overline{\Omega})} \leqslant C$ due to the regularity result (cf. [14]), where $C > 0$ is independent of m because u_m and u_s are solutions of (P) within $[0, u]$. Since the inclusion of $C^{1,\nu}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ is compact and $u_m \to u_s = 0$ in $W^{1,p}(\Omega)$, $u_m(x) \to 0$ uniformly in $x \in \overline{\Omega}$ holds. This yields a contradiction because $\|u_m\|_\infty > \delta_0$ for every m by Lemma 17. So, $u_s \not\equiv 0$, whence u_s is a positive solution of (P) with $\min_{\overline{\Omega}} u_s > 0$ by Remark 8(ii) (note $u \geqslant u_s \geqslant 0$ in Ω).

Finally, we shall prove the minimality property of u_s among the positive solutions. Let $w \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a positive solution of (P). Then, by Remark 8, we have $w \in C^{1,\nu}(\overline{\Omega})$ ($\nu \in (0, 1)$) and $\min_{\overline{\Omega}} w > 0$ (note that $(\widetilde{f0+})$ is weaker than $(f0+)$). So, there exists an $m \in \mathbb{N}$ satisfying $\varepsilon_m < \min_{\overline{\Omega}} w$. Then, it follows from Proposition 16(iii) that there exists a $v_m \in [\varepsilon_m, \min\{u_m, w\}]$ being the smallest solution of (P) in $[\varepsilon_m, \min\{u_m, w\}]$ (note that $\min\{u_m, w\}$ is a super-solution). By the minimality of u_m in $[\varepsilon_m, u]$ and since $[\varepsilon_m, \min\{u_m, w\}] \subset [\varepsilon_m, u]$, we have $u_s \leqslant u_m \leqslant v_m \leqslant \min\{u_m, w\} \leqslant w$ (note that $\{u_m\}$ is nonincreasing). This shows that u_s is a smallest positive solution in view of the arbitrariness of w. $\square$

The following result can be shown by the same argument as for the positive solution.

Proposition 20. *Assume* $(f0-)$. *If* (P) *has a sub-solution* $v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *with* $\operatorname{ess\,sup}_\Omega v < 0$, *then* (P) *has a largest negative solution* $u_l \in C^1(\overline{\Omega})$ *with* $u_l \geqslant v$ *in the sense that* $u_l \geqslant w$ *in* Ω *for every negative solution* $w \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *of* (P).

Remark 21. We point out the following:

(i) if u_s is the smallest positive solution of (P), then $K(I_{[0,u_s]}) = \{0, u_s\}$;
(ii) if u_l is the largest negative solution of (P), then $K(I_{[u_l,0]}) = \{0, u_l\}$;
(iii) assume that u_s and u_l are the smallest positive solution and the largest negative solution of (P), respectively. If $u \in K(I_{[u_l,u_s]}) \setminus \{0, u_s, u_l\}$, then u is a sign changing solution of (P),

where $K(\Psi) := \{u \in W^{1,p}(\Omega);\ \Psi'(u) = 0\}$ for a C^1 functional Ψ on $W^{1,p}(\Omega)$.

Proof. Since the above facts are important in the proofs of our results, we give the proofs of (i) and (iii).

(i): Let u be a non-trivial critical point of $I_{[0,u_s]}$. Then, by Lemma 14, $u \in C^1(\overline{\Omega})$ is a non-trivial solution of (P) with $0 \leqslant u \leqslant u_s$. Moreover, it follows from Remark 8(ii) that u is positive because of $u \neq 0$. Therefore, $u = u_s$ holds since u_s is the smallest positive solution.

(iii): Let u be a non-trivial critical point of $I_{[u_l,u_s]}$. Then, by Lemma 14, u is a non-trivial solution of (P) with $u_l \leqslant u \leqslant u_s$ in Ω. If $u \geqslant 0$ holds, then we have $u = u_s$ by the same argument as above. Similarly, if $u \leqslant 0$ holds, then we see $u = u_l$. So, u changes sign since u is different from u_l and u_s. $\square$

3. $C^1(\overline{\Omega})$ versus $W^{1,p}(\Omega)$ local minimizer

3.1. Coincidence of local minimizers

In this subsection, we state the relation between $C^1(\overline{\Omega})$-local minimizer and $W^{1,p}(\Omega)$-local minimizer for the Neumann problems.

Definition 22. We say that $u \in W^{1,p}(\Omega)$ is an X-local minimizer of I if there exists a $\rho > 0$ such that

$$I(u_0) \leqslant I(u_0 + h) \quad \text{for all } h \in X, \text{ with } \|h\|_X \leqslant \rho,$$

where I is the functional defined by (4) and X is a Banach space continuously embedded in $W^{1,p}(\Omega)$.

It is well known that in the case of Dirichlet boundary condition, a $C_0^1(\overline{\Omega})$-local minimizer becomes a $W_0^{1,p}(\Omega)$-local minimizer (see [6] for $p = 2$, [2]). Recently, the second author and Papageorgiou showed such a result concerning our problem. Here, we recall their result [17, Theorem 3.1]: Define

$$C_n^1(\overline{\Omega}) := \left\{u \in C^1(\overline{\Omega});\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\right\} \tag{15}$$

and denote the closure of $C_n^1(\overline{\Omega})$ in $W^{1,p}(\Omega)$ topology by $W_n^{1,p}(\Omega)$. Under the subcritical growth condition (g) (see Section 2), if $u_0 \in W_n^{1,p}(\Omega)$ is a $C_n^1(\overline{\Omega})$-local minimizer of I, then $u_0 \in C_n^1(\overline{\Omega})$ and it is a $W_n^{1,p}(\Omega)$-local minimizer of I.

In the next subsection, we prove that $W_n^{1,p}(\Omega) = W^{1,p}(\Omega)$ in the case where $\partial\Omega$ is of class C^3. Hence, the following result is derived from Theorem 27 and [17, Theorem 3.1] (refer to the proof of Proposition 24).

Theorem 23. *Suppose that Ω is a bounded domain in $\mathbb{R}^N$ with C^3 boundary $\partial\Omega$ and assume that f satisfies (g). If $u_0 \in W^{1,p}(\Omega)$ is a $C_n^1(\overline{\Omega})$-local minimizer of the functional I defined by* (4)*, then u_0 is also a $W^{1,p}(\Omega)$-local minimizer of I and it belongs to $C_n^1(\overline{\Omega})$.*

In the case of C^2 boundary $\partial\Omega$, we can obtain the following result concerning $C^1(\overline{\Omega})$ versus $W^{1,p}(\Omega)$ local minimizers by an argument of the same type as in [17, Theorem 3.1].

Proposition 24. *Suppose that Ω is a bounded domain in $\mathbb{R}^N$ with C^2 boundary $\partial\Omega$ and assume that f satisfies (g). If $u_0 \in W^{1,p}(\Omega)$ is a $C^1(\overline{\Omega})$-local minimizer of the functional I defined by* (4)*, then u_0 is also a $W^{1,p}(\Omega)$-local minimizer of I and it belongs to $C_n^1(\overline{\Omega})$.*

Proof. Let $u_0 \in W^{1,p}(\Omega)$ be a $C^1(\overline{\Omega})$-local minimizer of I. By a standard argument (cf. [15, Theorem 1.3]), we have $\langle I'(u_0), h\rangle = 0$ for every $h \in C^1(\overline{\Omega})$ (note that I is C^1 on $W^{1,p}(\Omega)$ under (g)). Since $C^1(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$, the equality above implies that u_0 is a critical point of I, and hence $u_0 \in C^{1,\nu}(\overline{\Omega})$ (for some $0 < \nu < 1$) (see Remark 8) is a solution for

$$-\operatorname{div} A(x, \nabla u_0) = f(x, u_0) \quad \text{in } \Omega, \qquad \frac{\partial u_0}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{16}$$

By way of contradiction, we assume that u_0 is not a $W^{1,p}(\Omega)$-local minimizer of I. Let $r \in (p, p^*)$ be a constant for which (g) is satisfied. By the Sobolev embedding theorem, we may suppose that for every $\varepsilon > 0$ there holds

$$m_\varepsilon := \inf\bigl\{I(u_0 + h);\ h \in \overline{B_r(0,\varepsilon)}\bigr\} < I(u_0), \tag{17}$$

where $\overline{B_r(0,\varepsilon)} := \{v \in W^{1,p}(\Omega);\ \|v\|_r \leqslant \varepsilon\}$. Note that a minimizing sequence for m_ε is bounded in $W^{1,p}(\Omega)$ because of the boundedness of it in $L^r(\Omega)$, (g) and (3) (refer also to (18)). Thus, for every $\varepsilon > 0$, m_ε is attained by some $h_\varepsilon \in \overline{B_r(0,\varepsilon)}$ since I is weakly lower semi-continuous on $W^{1,p}(\Omega)$ (see the proof of Lemma 11) and $\overline{B_r(0,\varepsilon)}$ is weakly closed in $W^{1,p}(\Omega)$. In addition, it is seen also that $\{h_\varepsilon\}$ is bounded in $W^{1,p}(\Omega)$ by the boundedness of $\|h_\varepsilon\|_r$, (g) and the following inequality:

$$\begin{aligned}\frac{C_0}{p(p-1)}\|\nabla u_0 + \nabla h_\varepsilon\|_p^p &\leqslant \int_\Omega G(x, \nabla u_0 + \nabla h_\varepsilon)\,dx \\ &= I(u_0 + h_\varepsilon) + \int_\Omega F(x, u_0 + h_\varepsilon)\,dx,\end{aligned} \tag{18}$$

where we use (3) in the first inequality. Here, we claim that there exists a $\lambda_\varepsilon \leqslant 0$ such that

$$-\operatorname{div} A(x, \nabla u_0 + \nabla h_\varepsilon) = f(x, u_0 + h_\varepsilon) + \lambda_\varepsilon |h_\varepsilon|^{r-2} h_\varepsilon \quad \text{in } \Omega. \tag{19}$$

Indeed, if $\|h_\varepsilon\|_r < \varepsilon$, then $u_0 + h_\varepsilon$ is a $W^{1,p}(\Omega)$-local minimizer of I with $I(u_0 + h_\varepsilon) = m_\varepsilon < I(u_0)$ by the continuity of the embedding from $W^{1,p}(\Omega)$ to $L^r(\Omega)$. So $u_0 + h_\varepsilon$ is a critical point of I, and hence (19) holds with $\lambda_\varepsilon = 0$. If $\|h_\varepsilon\|_r = \varepsilon$, by the Lagrange multiplier rule, there exists a $\lambda_\varepsilon \in \mathbb{R}$ satisfying (19). Using the inequality

$$0 \geqslant \lim_{t \to +0} \frac{I(u_0 + h_\varepsilon - t h_\varepsilon) - I(u_0 + h_\varepsilon)}{-t} = \langle I'(u_0 + h_\varepsilon), h_\varepsilon \rangle = \lambda_\varepsilon \|h_\varepsilon\|_r^r,$$

we have $\lambda_\varepsilon \leqslant 0$ since $I(u_0 + h_\varepsilon) = m_\varepsilon$ and $h_\varepsilon - t h_\varepsilon \in \overline{B_r(0, \varepsilon)}$ for $1 \geqslant t \geqslant 0$. Thus, our claim is shown.

Set $\tilde{A}(x, y) := A(x, \nabla u_0(x) + y) - A(x, \nabla u_0(x))$. Then, we see the following

$$-\operatorname{div} \tilde{A}(x, \nabla h_\varepsilon) = f(x, u_0 + h_\varepsilon) - f(x, u_0) + \lambda_\varepsilon |h_\varepsilon|^{r-2} h_\varepsilon \quad \text{in } \Omega \tag{20}$$

by (16) and (19). By the Moser iteration method (see Theorem C in Appendix A with $\mu = \lambda_\varepsilon$ and $h(x, u) = |u|^{r-2} u$), there exists an $M_1 > 0$ independent of ε such that $\|h_\varepsilon\|_\infty \leqslant M_1$ for every $\varepsilon > 0$ because f satisfies the subcritical growth condition and $\|h_\varepsilon\|$ is bounded (note $\lambda_\varepsilon \leqslant 0$). By applying the regularity result of [14] to the solution $u_0 + h_\varepsilon$ of (19), we have $u_0 + h_\varepsilon \in C^1(\overline{\Omega})$, and so $h_\varepsilon \in C^1(\overline{\Omega})$ holds for every $\varepsilon > 0$.

Now, we claim that there exists a $d_0 > 0$ such that $|\lambda_\varepsilon |h_\varepsilon(x)|^{r-2} h_\varepsilon(x)| \leqslant d_0$ for every $x \in \Omega$ and $\varepsilon > 0$. To simplify the notation, we set $f_0(x, u) := f(x, u_0(x) + u) - f(x, u_0(x))$. Because there exist positive constants d_1 and d_2 satisfying $|f_0(x, u)| \leqslant d_1 |u|^{r-1} + d_2$ for every $u \in \mathbb{R}$, a.e. $x \in \Omega$ by (g) and $u_0 \in C^1(\overline{\Omega})$, we have for every $\rho > 0$ the following inequality:

$$\left| f_0(x, u) \right| \leqslant \left(d_1 + d_2 \rho^{1-r} \right) |u|^{r-1} \quad \text{for all } |u| \geqslant \rho \text{ and a.e. } x \in \Omega. \tag{21}$$

Taking $(h_\varepsilon - \rho)_+$ as test function in (20) with $\rho > 0$, the following inequality follows from (20), (21), $\lambda_\varepsilon \leqslant 0$ and the monotonicity of A in the second variable (refer also to (42)):

$$\begin{aligned} 0 &\leqslant \int_{h_\varepsilon > \rho} \tilde{A}(x, \nabla h_\varepsilon) \nabla h_\varepsilon \, dx = \int_\Omega \tilde{A}(x, \nabla h_\varepsilon) \nabla (h_\varepsilon - \rho)_+ \, dx \\ &\leqslant \left(d_1 + d_2 \rho^{1-r} \right) \int_\Omega h_\varepsilon^{r-1} (h_\varepsilon - \rho)_+ \, dx - |\lambda_\varepsilon| \int_\Omega h_\varepsilon^{r-1} (h_\varepsilon - \rho)_+ \, dx. \end{aligned}$$

This yields $|\lambda_\varepsilon| \leqslant d_1 + d_2 \rho^{1-r}$ for $\rho > 0$ provided $(h_\varepsilon - \rho)_+ \not\equiv 0$. Similarly, in the case where $(h_\varepsilon + \rho)_- \not\equiv 0$, we also have $|\lambda_\varepsilon| \leqslant d_1 + d_2 \rho^{1-r}$. Therefore, by choosing $\rho_\varepsilon := \|h_\varepsilon\|_\infty / 2 > 0$ for each $\varepsilon > 0$, we can obtain

$$|\lambda_\varepsilon| \left| h_\varepsilon(x) \right|^{r-1} \leqslant |\lambda_\varepsilon| \|h_\varepsilon\|_\infty^{r-1} \leqslant d_1 \|h_\varepsilon\|_\infty^{r-1} + 2^{r-1} d_2 \leqslant d_1 M_1^{r-1} + 2^{r-1} d_2 \tag{22}$$

for every $x \in \Omega$ and $\varepsilon > 0$ (note $h_\varepsilon \in C^1(\overline{\Omega})$ and $\|h_\varepsilon\|_\infty \leqslant M_1$ for all $\varepsilon > 0$). So, our claim is shown.

Returning to the proof, if we set $f_\varepsilon(x,u) := f(x,u) + \lambda_\varepsilon |h_\varepsilon(x)|^{r-2} h_\varepsilon(x)$, then f_ε is bounded on $\Omega \times [-M_1 - \|u_0\|_\infty, M_1 + \|u_0\|_\infty]$ uniformly in $\varepsilon > 0$ by (22). By applying the regularity result of [14] to the solution $u_0 + h_\varepsilon$ for (19), that is, $-\operatorname{div} A(x, \nabla u_0 + \nabla h_\varepsilon) = f_\varepsilon(x, u_0 + h_\varepsilon)$ in Ω, there exist $\theta \in (0,1)$ and $M_2 > 0$ independent of ε such that $u_0 + h_\varepsilon \in C^{1,\theta}(\overline{\Omega})$ and $\|u_0 + h_\varepsilon\|_{C^{1,\theta}(\overline{\Omega})} \leqslant M_2$ for every $\varepsilon > 0$. Since $C^{1,\theta}(\overline{\Omega})$ is embedded compactly into $C^1(\overline{\Omega})$, we infer that $u_0 + h_\varepsilon \to u_0$ as $\varepsilon \downarrow 0$ in $C^1(\overline{\Omega})$ by noting that $h_\varepsilon \to 0$ in $L^r(\Omega)$ as $\varepsilon \downarrow 0$. Consequently, we get a contradiction by the following inequality

$$I(u_0 + h_\varepsilon) = m_\varepsilon < I(u_0) \leqslant I(u_0 + h_\varepsilon)$$

for sufficiently small $\varepsilon > 0$, where we use our assumption that u_0 is a $C^1(\overline{\Omega})$-local minimizer of I. This completes the proof. □

3.2. The density of $C_n^1(\overline{\Omega})$ in $W^{1,p}(\Omega)$

Lemma 25. *Let $\mathbb{R}_+^N := \{x = (x', x_N) \in \mathbb{R}^N;\ x' \in \mathbb{R}^{N-1},\ x_N > 0\}$ and suppose that $u \in W^{2,q}(\mathbb{R}_+^N)$ with $1 \leqslant q < \infty$. Then there exists a sequence $\{\varphi_m\}_m$ in $C^1(\overline{\mathbb{R}_+^N})$ having the following properties*:

(1) $\|u - \varphi_m\|_{W^{1,q}(\mathbb{R}_+^N)} \to 0$ $(m \to \infty)$,
(2) $\frac{\partial \varphi_m}{\partial x_N} = 0$ *on* $\mathbb{R}^{N-1} \times \{0\}$ *for every* $m \in \mathbb{N}$.

Moreover, for every $r > 0$, φ_m's can be chosen so that $\operatorname{supp} \varphi_m$ *is contained in the r-neighborhood of* $\operatorname{supp} u$.

Proof. Step 1. First of all, note that the function $u(x', 0)$ $(x' \in \mathbb{R}^{N-1})$ in the sense of the trace of u belongs to $W^{1,q}(\mathbb{R}^{N-1})$. Using this function and a $\rho \in C_0^\infty(\mathbb{R})$ with $\rho(t) = 1$ on $[-t_0, 0]$ and $\rho(t) = 0$ on $(-\infty, -2t_0]$ for some $t_0 > 0$, set

$$\tilde{u}(x', x_N) := \begin{cases} u(x', x_N) & (x_N > 0), \\ u(x', 0)\rho(x_N) & (x_N \leqslant 0). \end{cases}$$

Then it is clear that $\tilde{u}$ extends u and $\tilde{u} \in L^q(\mathbb{R}^N)$. Moreover, $\tilde{u} \in W^{1,q}(\mathbb{R}^N)$. Indeed, for $i = 1, 2, \ldots, N$ and $\varphi \in C_0^\infty(\mathbb{R}^N)$, it readily follows that

$$\begin{aligned} \int_{\mathbb{R}^N} \tilde{u}(x) \frac{\partial \varphi}{\partial x_i}(x)\, dx &= \int_{\mathbb{R}_+^N} u(x) \frac{\partial \varphi}{\partial x_i}(x)\, dx \\ &\quad + \int_{x_N < 0} \left[\int_{\mathbb{R}^{N-1}} u(x', 0) \frac{\partial \varphi}{\partial x_i}(x', x_N)\, dx' \right] \rho(x_N)\, dx_N \\ &\equiv \mathrm{I} + \mathrm{II}. \end{aligned} \tag{23}$$

Now, let $1 \leqslant i \leqslant N-1$. Then, as for term I, we can adopt the technique in Brezis [5, pp. 158–159]. Namely, we take a $\chi \in C^\infty(\mathbb{R})$ satisfying $\chi \equiv 0$ on $(-\infty, 1/2]$ and $\chi \equiv 1$ on $[1, \infty)$. Setting $\chi_\varepsilon(t) := \chi(t/\varepsilon)$ for $t \in \mathbb{R}$ and $\varepsilon > 0$, we obtain $\varphi(x)\chi_\varepsilon(x_N)|_{\mathbb{R}_+^N} \in C_0^\infty(\mathbb{R}_+^N)$. Hence

$$\int_{\mathbb{R}^N_+} u(x)\frac{\partial}{\partial x_i}\bigl(\varphi(x)\chi_\varepsilon(x_N)\bigr)\,dx = -\int_{\mathbb{R}^N_+} \frac{\partial u}{\partial x_i}(x)\varphi(x)\chi_\varepsilon(x_N)\,dx.$$

Since $(\partial/\partial x_i)(\varphi(x)\chi_\varepsilon(x_N)) = (\partial\varphi/\partial x_i)(x)\chi_\varepsilon(x_N)$ (note $i \neq N$), by letting $\varepsilon \downarrow 0$, we obtain

$$\mathrm{I} = \int_{\mathbb{R}^N_+} u(x)\frac{\partial\varphi}{\partial x_i}(x)\,dx = -\int_{\mathbb{R}^N_+} \frac{\partial u}{\partial x_i}(x)\varphi(x)\,dx. \tag{24}$$

On the other hand, $u(x',0) \in W^{1,q}(\mathbb{R}^{N-1})$ implies

$$\begin{aligned}
\mathrm{II} &= -\int_{x_N<0}\left[\int_{\mathbb{R}^{N-1}} \frac{\partial u(x',0)}{\partial x_i}\varphi\bigl(x',x_N\bigr)\,dx'\right]\rho(x_N)\,dx_N \\
&= -\int_{\mathbb{R}^N_-}\left(\frac{\partial u(x',0)}{\partial x_i}\rho(x_N)\right)\varphi\bigl(x',x_N\bigr)\,dx'\,dx_N, \qquad (25)
\end{aligned}$$

where $\mathbb{R}^N_- := \{(x',x_N);\ \ x_N < 0\}$. Recalling that $u \in W^{2,q}(\mathbb{R}^N_+)$ and hence $\partial u(x',0)/\partial x_i \in L^q(\mathbb{R}^{N-1})$, we realize that (23), (24) and (25) yield $\partial\tilde{u}/\partial x_i \in L^q(\mathbb{R}^N)$ in the sense of distributions.

The remaining case of $i = N$ in (23) is more delicate. To treat this case, let us recall the following fact: (up to the equivalence of a.e.-equality) $u(x',x_N)$ may be supposed to be absolutely continuous in x_N for almost every $x' \in \mathbb{R}^{N-1}$ with x_N-derivative being $\partial u(x',x_N)/\partial x_N$ for a.e. x_N (Ziemer [22, Theorem 2.1.4]). We can also assume that $\lim_{x_N\downarrow 0} u(x',x_N) = u(x',0)$ (= the trace) for a.e. x' since $\int_0^\infty |\partial u(x',x_N)/\partial x_N|^q\,dx_N < \infty$ for a.e. x'. Therefore, term I of (23) can be transformed as follows:

$$\begin{aligned}
\mathrm{I} &= \int_{\mathbb{R}^{N-1}}\left\{\int_0^\infty u\bigl(x',x_N\bigr)\frac{\partial\varphi}{\partial x_N}\bigl(x',x_N\bigr)\,dx_N\right\}dx' \\
&= \int_{\mathbb{R}^{N-1}}\left\{-u\bigl(x',0\bigr)\varphi\bigl(x',0\bigr) - \int_0^\infty \frac{\partial u}{\partial x_N}\bigl(x',x_N\bigr)\varphi\bigl(x',x_N\bigr)\,dx_N\right\}dx' \\
&= -\int_{\mathbb{R}^{N-1}} u\bigl(x',0\bigr)\varphi\bigl(x',0\bigr)\,dx' - \int_{\mathbb{R}^N_+}\frac{\partial u}{\partial x_N}(x)\varphi(x)\,dx. \qquad (26)
\end{aligned}$$

On the other hand,

$$\mathrm{II} = \int_{\mathbb{R}^N_-} u\bigl(x',0\bigr)\rho(x_N)\frac{\partial\varphi}{\partial x_N}(x)\,dx$$

$$= \int_{\mathbb{R}^{N-1}} u(x',0)\left\{\int_{-\infty}^{0} \rho(x_N)\frac{\partial\varphi}{\partial x_N}(x',x_N)\,dx_N\right\}dx'$$

$$= \int_{\mathbb{R}^{N-1}} u(x',0)\varphi(x',0)\,dx' - \int_{\mathbb{R}^N_-} u(x',0)\rho'(x_N)\varphi(x)\,dx \tag{27}$$

holds, where ρ' means the usual derivative of ρ. So, if we define

$$v(x',x_N) := \begin{cases} \frac{\partial u}{\partial x_N}(x',x_N) & (x_N > 0), \\ u(x',0)\rho'(x_N) & (x_N < 0), \end{cases}$$

then $v \in L^q(\mathbb{R}^N)$ and (26), (27) yield

$$\int_{\mathbb{R}^N} \tilde{u}(x)\frac{\partial\varphi}{\partial x_N}(x)\,dx = -\int_{\mathbb{R}^N} v(x)\varphi(x)\,dx.$$

Thus we have proved $\tilde{u} \in W^{1,q}(\mathbb{R}^N)$.

Step 2. For every $h \in \mathbb{R}$ and $w : \mathbb{R}^N \to \mathbb{R}$, let $(\tau_h w)(x', x_N) := w(x', x_N - h)$ be the translation of w. Then, it is well known that $\tau_h w$ converges to w in $L^q(\mathbb{R}^N)$ (resp. $W^{1,q}(\mathbb{R}^N)$) if $w \in L^q(\mathbb{R}^N)$ (resp. $W^{1,q}(\mathbb{R}^N)$). In addition to translation, we use usual mollifier $\{\rho_\varepsilon\}_{\varepsilon>0}$ for which we assume $\operatorname{supp}\rho_\varepsilon \subset \{x \in \mathbb{R}^N;\ |x| < \varepsilon\}$ and $\|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)} = 1$ for every $\varepsilon > 0$. Clearly, $\rho_\varepsilon * (\tau_h w) = \tau_h(\rho_\varepsilon * w)$ holds for $w \in L^q(\mathbb{R}^N)$, $\varepsilon > 0$ and $h \in \mathbb{R}$. It is also well known and easily verified that $\rho_\varepsilon * (\tau_h w) \in C^\infty(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ for every $\varepsilon > 0$ and $h \in \mathbb{R}$ provided $w \in W^{1,q}(\mathbb{R}^N)$. Moreover,

$$\frac{\partial}{\partial x_i}\big(\rho_\varepsilon * (\tau_h w)\big) = \rho_\varepsilon * \left(\frac{\partial}{\partial x_i}(\tau_h w)\right) = \rho_\varepsilon * \left(\tau_h \frac{\partial w}{\partial x_i}\right) \tag{28}$$

holds for $i = 1, 2, \dots, N$ under the same conditions.

Next let us verify the following assertion: For $w \in W^{1,q}(\mathbb{R}^N)$ and $m \in \mathbb{N}$, set $\psi_m := \rho_{1/m} * (\tau_{1/m} w)$. Then ψ_m converges to w in $W^{1,q}(\mathbb{R}^N)$. For the verification of $\|\psi_m - w\|_{L^q(\mathbb{R}^N)} \to 0$ $(m \to \infty)$ note the following inequality:

$$\begin{aligned} \|\psi_m - w\|_{L^q(\mathbb{R}^N)} &\leqslant \big\|\rho_{1/m} * (\tau_{1/m} w) - \tau_{1/m} w\big\|_{L^q(\mathbb{R}^N)} + \|\tau_{1/m} w - w\|_{L^q(\mathbb{R}^N)} \\ &= \big\|\tau_{1/m}(\rho_{1/m} * w - w)\big\|_{L^q(\mathbb{R}^N)} + \|\tau_{1/m} w - w\|_{L^q(\mathbb{R}^N)} \\ &= \|\rho_{1/m} * w - w\|_{L^q(\mathbb{R}^N)} + \|\tau_{1/m} w - w\|_{L^q(\mathbb{R}^N)}. \end{aligned} \tag{29}$$

To show $\|\partial\psi_m/\partial x_i - \partial w/\partial x_i\|_{L^q(\mathbb{R}^N)} \to 0$ $(m \to \infty)$, note that inequality (29) is valid for $\partial w/\partial x_i \in L^q(\mathbb{R}^N)$ and hence we obtain

$$\left\|\rho_{1/m} * \left(\tau_{1/m}\frac{\partial w}{\partial x_i}\right) - \frac{\partial w}{\partial x_i}\right\|_{L^q(\mathbb{R}^N)} \to 0 \quad (m \to \infty).$$

Combining this with (28), we obtain the desired convergence.

Step 3. Now, let us return to our $\tilde{u}$ constructed in Step 1. By the result in Step 2, $\psi_m := \rho_{1/m} * (\tau_{1/m}\tilde{u})$ $(m \in \mathbb{N})$ belongs to $C^\infty(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ and converges to $\tilde{u}$ in $W^{1,q}(\mathbb{R}^N)$ as $m \to \infty$, hence $\varphi_m := \psi_m|_{\mathbb{R}^N_+}$ converges to u in $W^{1,q}(\mathbb{R}^N_+)$. If we show $(\partial\psi_m/\partial x_N)(x', 0) = 0$ for every $x' \in \mathbb{R}^{N-1}$, we are done. To prove this assertion, note that $\tau_h\tilde{u}(x', x_N) = u(x', 0)$ if $-t_0 < x_N < h$. Therefore, $(\partial\tau_{1/m}\tilde{u}/\partial x_N)(x', x_N) = 0$ provided $-t_0 < x_N < 1/m$. Taking this fact into account, we can see that (28) and the fact $\operatorname{supp}\rho_{1/m} \subset \{x;\ |x| < 1/m\}$ imply $(\partial\psi_m/\partial x_N)(x', 0) = 0$ for every $x' \in \mathbb{R}^{N-1}$.

Step 4. The last assertion concerning the support of the approximating functions is easily verified. In fact, for an $r > 0$, by choosing a sufficiently small $t_0 > 0$, the support of the function φ_m in Step 3 is contained in the r-neighborhood of $\operatorname{supp} u$ for sufficiently large m. □

Lemma 26. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with C^1-class boundary $\partial\Omega$, and let $1 \leqslant q < \infty$. Then, $W^{2,q}(\Omega)$ is dense in $W^{1,q}(\Omega)$.*

Proof. By the assumption that $\partial\Omega$ is of C^1-class, there exists an extension operator $E : W^{1,q}(\Omega) \to W^{1,q}(\mathbb{R}^N)$ that is bounded linear and satisfies $Eu|_\Omega = u$ for every $u \in W^{1,q}(\Omega)$ (see e.g., [5, Théorème IX.7]). Since $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{1,q}(\mathbb{R}^N)$, given a $u \in W^{1,q}(\Omega)$, there exists a sequence $\{\varphi_m\}_m$ in $C_0^\infty(\mathbb{R}^N)$ such that $\|Eu - \varphi_m\|_{W^{1,q}(\mathbb{R}^N)} \to 0$ as $m \to \infty$. It is clear that

$$\left(\frac{\partial}{\partial x_i} Eu\right)\Bigg|_\Omega = \frac{\partial}{\partial x_i} u \quad (1 \leqslant i \leqslant N)$$

holds, hence we obtain $\|u - \varphi_m|_\Omega\|_{W^{1,q}(\Omega)} \leqslant \|Eu - \varphi_m\|_{W^{1,q}(\mathbb{R}^N)}$. Noting that $\varphi_m|_\Omega \in W^{2,q}(\Omega)$, we see that the conclusion of this lemma holds. □

Theorem 27. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with C^3-class boundary $\partial\Omega$, and let $1 \leqslant q < \infty$. Then, $C_n^1(\overline{\Omega})$ is dense in $W^{1,q}(\Omega)$, where $C_n^1(\overline{\Omega})$ is the Banach space defined by* (15).

Proof. Step 1. Let $n(x)$ denote the unit outer normal of Ω at $x \in \partial\Omega$. Then it is a standard fact of differential geometry [3, Theorem 2.7.12] that there exists a positive number r_0 such that the mapping $\Phi : (x, r) \mapsto x + rn(x)$ gives a diffeomorphism from $\partial\Omega \times (-r_0, r_0)$ onto $\{x \in \mathbb{R}^N;\ \operatorname{dist}(x, \partial\Omega) < r_0\}$. Note that this Φ is of at least C^2-class since we are treating C^3-class $\partial\Omega$. From this fact, we can easily see that every $x \in \partial\Omega$ has an open neighborhood U_x that has the following properties:

(1) There exist an open neighborhood V_x of $\overline{U_x}$ and a C^2-class diffeomorphism Φ_x from V_x onto a neighborhood of $Q := \{x = (x', x_N) \in \mathbb{R}^N;\ |x'| < 1,\ |x_N| < 1\}$.
(2) $\Phi_x(U_x \cap \Omega) = Q_+ := \{(x', x_N) \in Q;\ x_N > 0\}$ and $\Phi_x(U_x \cap \partial\Omega) = Q_0 := \{(x', x_N) \in Q;\ x_N = 0\}$ hold.
(3) For every $(x', x_N) \in Q$, $\Phi_x^{-1}(x', x_N)$ always lies on the normal at $\Phi_x^{-1}(x', 0) \in \partial\Omega$.

By the compactness of $\partial\Omega$, there exists a finite set of points $\{p_1, p_2, \ldots, p_l\}$ in $\partial\Omega$ for which $\partial\Omega \subset \bigcup_{i=1}^l U_{p_i}$ holds. For the sake of convenience, hereafter we write U_i and Φ_i instead of U_{p_i} and Φ_{p_i}, respectively. We can also take an open set U_0 with the properties $\Omega \setminus \bigcup_{i=1}^l U_i \subset U_0$ and $\overline{U_0} \subset \Omega$. Let $\{\rho_i\}_{i=0}^l$ be a C^∞ class partition of unity subordinate to the covering $\{U_0, U_1, \ldots, U_l\}$ of $\overline{\Omega}$. Namely, $\{\rho_i\}_{i=0}^l$ has the following properties:

(a) $\rho_i \in C^\infty(\mathbb{R}^N)$ and $\rho_i \geqslant 0$ for $i = 0, 1, \dots, l$.
(b) $\operatorname{supp} \rho_i \subset U_i$ $(0 \leqslant i \leqslant l)$ and $\sum_{i=0}^{l} \rho_i = 1$ in a neighborhood of $\overline{\Omega}$.

Step 2. First note that if we can show that the closure $W_n^{1,q}(\Omega)$ of $C_n^1(\overline{\Omega})$ in $W^{1,q}(\Omega)$ contains $W^{2,q}(\Omega)$, then the conclusion of the theorem is obtained by Lemma 26.

Now suppose a function $u \in W^{2,q}(\Omega)$ is given. u is rewritten as the sum $\sum_{i=0}^{l} \rho_i u$ and clearly $\rho_0 u \in W_n^{1,q}(\Omega)$. So, if we can show that $\rho_i u \in W_n^{1,q}(\Omega)$ for every $i = 1, 2, \dots, l$, the proof of the theorem is completed.

For $1 \leqslant i \leqslant l$, $\rho_i u \in W^{2,q}(\Omega)$ and $\operatorname{dist}(\operatorname{supp} \rho_i u, \partial U_i) > 0$. Therefore $v_i := (\rho_i u) \circ \Phi_i^{-1} \in W^{2,q}(Q_+)$ (recall that Φ_i is a C^2-diffeomorphism on an open neighborhood of $\overline{U_i}$) and $\operatorname{dist}(\operatorname{supp} v_i, \partial Q_+ \setminus Q_0) > 0$. Hence the extension $\overline{v_i}$ of v_i to the whole of $\mathbb{R}_+^N$ that has value 0 on $\mathbb{R}_+^N \setminus Q_+$ belongs to $W^{2,q}(\mathbb{R}_+^N)$. Accordingly, because of Lemma 25, there exists a sequence $\{\varphi_m^i\}_m$ in $C^1(\overline{\mathbb{R}_+^N}) \cap W^{1,q}(\mathbb{R}_+^N)$ such that $\|\overline{v_i} - \varphi_m^i\|_{W^{1,q}(\mathbb{R}_+^N)} \to 0$ $(m \to \infty)$, $\operatorname{supp} \varphi_m^i \cap (\partial Q_+ \setminus Q_0) = \emptyset$ and $\partial \varphi_m^i / \partial x_N = 0$ on $\mathbb{R}^{N-1} \times \{0\}$. Then, $\|\rho_i u - \varphi_m^i \circ \Phi_i\|_{W^{1,q}(U_i \cap \Omega)} \to 0$ $(m \to \infty)$ holds since the composition via Φ_i gives an isomorphism between $W^{1,q}(Q_+)$ and $W^{1,q}(U_i \cap \Omega)$. Moreover, property (3) of Φ_i implies that the normal derivative of $\varphi_m^i \circ \Phi_i$ vanishes on $U_i \cap \partial\Omega$. Since $\varphi_m^i \circ \Phi_i$ is zero in a neighborhood of $\partial U_i \cap \Omega$, it may be considered as an element of $C^1(\overline{\Omega})$ and as such a function $\varphi_m^i \circ \Phi_i \in C^1(\overline{\Omega}) \cap W^{1,q}(\Omega)$, $\|\rho_i u - \varphi_m^i \circ \Phi_i\|_{W^{1,q}(\Omega)} \to 0$ and $(\partial/\partial n)(\varphi_m^i \circ \Phi_i) = 0$ on $\partial\Omega$. Hence $\rho_i u \in W_n^{1,q}(\Omega)$ and the proof is completed. □

4. The proofs of the theorems

Proof of Theorem 1. By $(f+)$, we see that the constant function $T^+ > 0$ is a super-solution of (P) because $-\operatorname{div} A(x, \nabla T^+) = 0 \geqslant f(x, T^+)$ in Ω. It follows from Proposition 19 that there exists the smallest positive solution $u_s \in C^1(\overline{\Omega})$ with $T^+ \geqslant \max_{\overline{\Omega}} u_s > \delta_0$ (see Lemma 17). Similarly, by using $(f-)$ and Proposition 20, we obtain the largest negative solution $u_l \in C^1(\overline{\Omega})$ with $-\delta_1 > \min_{\overline{\Omega}} u_l \geqslant T^-$ (see Lemma 18). To show the existence of a sign-changing solution, it suffices to obtain a non-trivial critical point of $I_{[u_l,u_s]}$ different from u_s and u_l (see Remark 21 and Section 2.1 for the definition of the functional). For this reason, we may assume that $I_{[u_l,u_s]}$ has only finitely many critical points.

Recall that $I_{[0,u_s]}$, $I_{[u_l,0]}$ and $I_{[u_l,u_s]}$ satisfy the Palais–Smale condition since the operator V has the $(S)_+$ property and all functionals above are coercive by Lemma 11.

Now, we prove that u_s is the unique global minimizer of $I_{[0,u_s]}$ and in addition that $I_{[0,u_s]}(u_s) < 0$. Indeed, by $(F0)$, we have

$$I_{[0,u_s]}(\delta) = -\int_{\Omega} F_{[0,u_s]}(x, \delta)\, dx = -\int_{\Omega} F(x, \delta)\, dx \leqslant -\frac{\alpha C_1 \delta^p \mu(\Omega)}{p(p-1)} < 0$$

for $0 < \delta < \min\{\delta_4, \min_{\overline{\Omega}} u_s\}$ (note $\alpha > 0$), which shows that $\inf_{W^{1,p}(\Omega)} I_{[0,u_s]} < 0$. Since $I_{[0,u_s]}$ has a global minimizer by Lemma 11, u_s is the unique global minimizer (see Remark 21). Similarly, we can show that u_l is the unique global minimizer of $I_{[u_l,0]}$ and satisfies that $I_{[u_l,0]}(u_l) < 0$. By noting that $I_{[u_l,u_s]} = I_{[0,u_s]}$ on $\{u \in W^{1,p}(\Omega);\ u(x) \geqslant 0 \text{ for a.e. } x \in \Omega\}$ and $I_{[u_l,u_s]} = I_{[u_l,0]}$ on $\{u \in W^{1,p}(\Omega);\ u(x) \leqslant 0 \text{ for a.e. } x \in \Omega\}$, we know that u_s and u_l are $C^1(\overline{\Omega})$-local minimizers of $I_{[u_l,u_s]}$ because $\min_{\overline{\Omega}} u_s > 0$ and $\max_{\overline{\Omega}} u_l < 0$ (see Remark 8). We can apply Proposition 24 to $I_{[u_l,u_s]}$ since the nonlinearity $f_{[u_l,u_s]}(x,t) + p(t - u_s(x))_+^{p-1} - p(t - u_l(x))_-^{p-1}$ satisfies the subcritical growth conditions (g) (note also that f is bounded on $\Omega \times [\min_{\overline{\Omega}} u_l, \max_{\overline{\Omega}} u_s]$ and so $f_{[u_l,u_s]}$ is bounded on $\Omega \times \mathbb{R}$). Therefore, u_s and u_l are $W^{1,p}(\Omega)$-local minimizers of $I_{[u_l,u_s]}$ by Proposition 24. At this point, we may suppose that either u_s or u_l is the global minimizer of $I_{[u_l,u_s]}$ (Lemma 11 guarantees its existence). In fact, if it is not the case, then there exists a global minimizer of $I_{[u_l,u_s]}$ other than u_l and u_s, which is a sign-changing solution of (P) (see Remark 21 and note $\min_{W^{1,p}(\Omega)} I_{[u_l,u_s]} < 0$). So, let u_l be a global minimizer (in the case of u_s, we can proceed in the same way). Hence, because $I_{[u_l,u_s]}$ satisfies the Palais–Smale condition and we have assumed that $I_{[u_l,u_s]}$ has only finitely many critical points, we may suppose that there exists a positive constant $r < \|u_l - u_s\|$ such that

$$\max\bigl\{I_{[u_l,u_s]}(u_l), I_{[u_l,u_s]}(u_s)\bigr\} = I_{[u_l,u_s]}(u_s) < \inf\bigl\{I_{[u_l,u_s]}(v);\ \|v - u_s\| = r\bigr\}, \tag{30}$$

that is, $I_{[u_l,u_s]}$ has the mountain pass geometry.

Let us construct a path γ_0 starting from u_s to u_l such that $0 \notin \gamma_0([0,1])$ and $I_{[u_l,u_s]}(\gamma_0(t)) \leqslant 0$ for every $t \in [0,1]$. Since $(\alpha,\beta) \in \mathscr{C}$ as in $(F0)$, there exists a non-trivial solution $w \in C^{1,\nu}(\overline{\Omega})$ (for some $0 < \nu < 1$) of $-\Delta_p w = \alpha w_+^{p-1} - \beta w_-^{p-1}$ in Ω such that $\|w\|_\infty < \min\{\delta_4, \min_{\overline{\Omega}} u_s, \min_{\overline{\Omega}} |u_l|\}$ (see Remark 6 for details). Then, by $(F0)$, we have

$$\begin{aligned} I_{[u_l,u_s]}\bigl(tw_+ - (1-t)w_-\bigr) &\leqslant \frac{t^p C_1}{p(p-1)}\|\nabla w_+\|_p^p + \frac{(1-t)^p C_1}{p(p-1)}\|\nabla w_-\|_p^p \\ &\quad - \frac{t^p \alpha C_1}{p(p-1)}\|w_+\|_p^p - \frac{(1-t)^p \beta C_1}{p(p-1)}\|w_-\|_p^p = 0 \end{aligned}$$

for every $t \in [0,1]$, where we use (3), $\|\nabla w_+\|_p^p = \alpha\|w_+\|_p^p$ and $\|\nabla w_-\|_p^p = \beta\|w_-\|_p^p$. Here, we recall that w changes sign (see [19, Proposition 2]). Since $w_+ \neq 0$ is not a critical point of $I_{[0,u_s]}$, $I_{[0,u_s]}(w_+) = I_{[u_l,u_s]}(w_+) \leqslant 0$ and u_s is the unique global minimizer of $I_{[0,u_s]}$, by applying the second deformation lemma (cf. [11, Theorem 5.2]) to $I_{[0,u_s]}$, we can obtain an $\eta_1 \in C([0,1], W^{1,p}(\Omega))$ such that $\eta_1(0) = w_+$, $\eta_1(1) = u_s$, $\eta_1(t) \neq u_s$ for every $t \neq 1$ and

$$I_{[0,u_s]}\bigl(\eta_1(t)\bigr) < I_{[0,u_s]}\bigl(\eta_1(0)\bigr) = I_{[0,u_s]}(w_+) \leqslant 0 \tag{31}$$

for every $t \in (0,1]$. Similarly, by applying the second deformation lemma to $I_{[u_l,0]}$, we have an $\eta_2 \in C([0,1], W^{1,p}(\Omega))$ such that $\eta_2(0) = -w_-$, $\eta_2(1) = u_l$, $\eta_2(t) \neq u_l$ for every $t \neq 1$ and

$$I_{[u_l,0]}\bigl(\eta_2(t)\bigr) < I_{[u_l,0]}\bigl(\eta_2(0)\bigr) = I_{[u_l,0]}(-w_-) = I_{[u_l,u_s]}(-w_-) \leqslant 0 \tag{32}$$

for every $t \in (0,1]$. Now, we define a path γ_0 by

$$\gamma_0(t) := \begin{cases} \eta_1(1-3t)_+ & \text{if } 0 \leqslant t \leqslant 1/3, \\ (2-3t)w_+ - (3t-1)w_- & \text{if } 1/3 \leqslant t \leqslant 2/3, \\ -\eta_2(3t-2)_- & \text{if } 2/3 \leqslant t \leqslant 1. \end{cases}$$

Here, we note that

$$\begin{aligned} I_{[u_l,u_s]}\bigl(\eta_1(t)_+\bigr) &= I_{[0,u_s]}\bigl(\eta_1(t)_+\bigr) \leqslant I_{[0,u_s]}\bigl(\eta_1(t)\bigr) < 0, \\ I_{[u_l,u_s]}\bigl(-\eta_2(t)_-\bigr) &= I_{[u_l,0]}\bigl(-\eta_2(t)_-\bigr) \leqslant I_{[u_l,0]}\bigl(\eta_2(t)\bigr) < 0 \end{aligned}$$

for every $t \in (0,1]$ by (31) and (32) (note that G is nonnegative). Therefore, $0 \notin \gamma_0([0,1])$ (note also $w_\pm \neq 0$) and $\max_{t\in[0,1]} I_{[u_l,u_s]}(\gamma_0(t)) \leqslant 0$ hold.

If $\gamma_0((0,1))$ contains at least one critical point of $I_{[u_l,u_s]}$, namely, there exists a $t_0 \in (0,1)$ such that $u_0 := \gamma_0(t_0)$ is a critical point of $I_{[u_l,u_s]}$, then u_0 is a sign changing solution of (P) because u_0 is different from 0, u_s and u_l. So, we may assume that $\gamma_0((0,1))$ contains no critical points of $I_{[u_l,u_s]}$. Because we are assuming that $I_{[u_l,u_s]}$ has only finitely many critical points, there exists an $\varepsilon_0 < 0$ such that

$$\varepsilon_0 > \max\bigl\{I_{[u_l,u_s]}(u_s), I_{[u_l,u_s]}(u_l)\bigr\} \tag{33}$$

and $I_{[u_l,u_s]}$ has no critical values in $[\varepsilon_0, 0)$. Then, by the second deformation lemma, there exists an $\eta \in C([0,1] \times I^0_{[u_l,u_s]} \setminus K_0, W^{1,p}(\Omega))$, where $I^d_{[u_l,u_s]} := \{v \in W^{1,p}(\Omega);\ I_{[u_l,u_s]}(v) \leqslant d\}$ for $d \in \mathbb{R}$ and $K_0 := \{v \in W^{1,p}(\Omega);\ I'_{[u_l,u_s]}(v) = 0$ and $I_{[u_l,u_s]}(v) = 0\}$, satisfying the following:

$$\begin{cases} \eta(0,u) = u \quad \text{for every } u \in I^0_{[u_l,u_s]} \setminus K_0, \\ \eta(t,u) = u \quad \text{for every } u \in I^{\varepsilon_0}_{[u_l,u_s]},\ t \in [0,1], \\ I_{[u_l,u_s]}\bigl(\eta(t,u)\bigr) \text{ is nonincreasing in } t \quad \text{for every } u \in I^0_{[u_l,u_s]} \setminus K_0, \\ I_{[u_l,u_s]}\bigl(\eta(1,u)\bigr) \leqslant \varepsilon_0 \quad \text{for every } u \in I^0_{[u_l,u_s]} \setminus K_0, \end{cases}$$

that is, $I^{\varepsilon_0}_{[u_l,u_s]}$ is a strong deformation retract of $I^0_{[u_l,u_s]} \setminus K_0$.

Now, we apply the mountain pass theorem to $I_{[u_l,u_s]}$ by defining

$$\Gamma := \bigl\{\gamma \in C\bigl([0,1], W^{1,p}(\Omega)\bigr);\ \gamma(0) = u_s,\ \gamma(1) = u_l\bigr\},$$
$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_{[u_l,u_s]}\bigl(\gamma(t)\bigr).$$

Here, $\eta(1,\gamma_0(\cdot)) \in \Gamma \cap I^{\varepsilon_0}_{[u_l,u_s]}$ (note (33)) yields $c \leqslant \varepsilon_0 < 0$ and (30) implies $\max\{I_{[u_l,u_s]}(u_l), I_{[u_l,u_s]}(u_s)\} < c$. Hence, the mountain pass theorem guarantees that c is a critical value since $I_{[u_l,u_s]}$ satisfies the Palais–Smale condition. Consequently, we can get a non-trivial critical point of $I_{[u_l,u_s]}$ different from u_s and u_l. So, the proof is completed. □

Proof of Theorem 2. The constant function $T^- < 0$ is a sub-solution of (P) by $(f-)$. By Lemma 11, we already know that $I_{[T^-,0]}$ has a global minimizer. In addition, we have

$$I_{[T^-,0]}(-\delta) = -\int_\Omega F_{[T^-,0]}(x,-\delta)\,dx \leqslant -\beta\delta^p\mu(\Omega)C_1/p(p-1) < 0$$

for $0 < \delta < \min\{\delta_4, |T^-|\}$ by $(F0)$, where δ_4 is the positive constant in $(F0)$. It turns out that $\min_{W^{1,p}(\Omega)} I_{[T^-,0]} < 0$. Thus, there exists a global minimizer $0 \neq u_0 \in W^{1,p}(\Omega)$ of $I_{[T^-,0]}$. It follows from Lemma 14 and Remark 8(iii) that u_0 is a negative solution of (P) belonging to $C^{1,\nu}(\overline{\Omega})$ for some $\nu \in (0,1)$ with $u_0 \geqslant T^-$ and $I_{[T^-,0]}(u_0) = \min_{W^{1,p}(\Omega)} I_{[T^-,0]} < 0$. Similarly, there exists a positive solution u_1 of (P) satisfying $I_{[0,T^+]}(u_1) = \min_{W^{1,p}(\Omega)} I_{[0,T^+]} < 0$ because $T^+ > 0$ is a super-solution of (P). So, in view of Lemma 14, to obtain another non-trivial solution, we may assume that $I_{[T^-,0]}$ and $I_{[0,T^+]}$ have no non-trivial critical points other than u_0 or u_1, respectively. Moreover, taking into account Lemma 14 to get another non-trivial solution, it is sufficient to show that $I_{[T^-,T^+]}$ has a non-trivial critical point different from u_0 and u_1. By the same argument as in the proof of Theorem 1, we see that both u_0 and u_1 are $W^{1,p}(\Omega)$-local minimizers of $I_{[T^-,T^+]}$ since $\min_{\overline{\Omega}} u_1 > 0$ and $\max_{\overline{\Omega}} u_0 < 0$ by $(\widetilde{f0\pm})$ (see Remark 8). Recall that all the functionals $I_{[T^-,0]}$, $I_{[0,T^+]}$ and $I_{[T^-,T^+]}$ satisfy the Palais–Smale condition because they are coercive by Lemma 11 and the operator V defined in Proposition 10 has the $(S)_+$ property. Consequently, we can apply the same argument as in the proof of Theorem 1 by replacing u_l and u_s with u_0 and u_1. Thus, by using the functionals $I_{[T^-,T^+]}$, $I_{[T^-,0]}$ and $I_{[0,T^+]}$ instead of $I_{[u_l,u_s]}$, $I_{[u_l,0]}$ and $I_{[0,u_s]}$ respectively, as in the proof of Theorem 1, we can show the existence of another non-trivial critical point of $I_{[T^-,T^+]}$. □

Proof of Theorem 4. By the same argument as in the proof of Theorem 1, we have a smallest positive solution $u_s \in C^1(\overline{\Omega})$ of (P) which is also a unique global minimizer of $I_{[0,u_s]}$ with $I_{[0,u_s]}(u_s) < 0$. Here, we note that $f_{[0,u_s]}$, $f_{[-\infty,u_s]}$ and $f_{[-\infty,0]}$ satisfy the subcritical growth condition (g). This fact implies that any solution of $-\operatorname{div} A(x,\nabla u) = f_{[v,w]}(x,u)$ in Ω, $\partial u/\partial n = 0$ on $\partial\Omega$ (where $[v,w] = [0,u_s]$ or $[-\infty,u_s]$ or $[-\infty,0]$) belongs to $C^{1,\nu}(\overline{\Omega})$ $(0 < \nu < 1)$ by $L^\infty(\Omega)$ estimates (see Theorem C in Appendix A) and the regularity result (cf. [14]).

Now, we shall prove this theorem by examining the following two cases:

(a) (P) has at least one negative solution;
(b) (P) has no negative solutions.

Case (a): Let $u \in W^{1,p}(\Omega)$ be a negative solution of (P). Then, since $u \in C^{1,\nu}(\overline{\Omega})$ and $\max_{\overline{\Omega}} u < 0$ by Remark 8 (note that $f = f_{[-\infty,0]}$ on $\Omega \times (-\infty,0]$), it follows from Proposition 20 that (P) has a largest negative solution u_l. Hence, the same argument as in Theorem 1 leads to the desired conclusion.

Case (b): In this case, we notice that $I_{[-\infty,0]}$ has no non-trivial critical points (refer to Lemma 15 and Remark 8). Combining this fact and $I_{[-\infty,0]}(-\delta) < 0$ for $0 < \delta < \delta_4$, we see that $I_{[-\infty,0]}$ has no global minimizer since $I_{[-\infty,0]}$ satisfies the Palais–Smale condition by Lemma 12, that is, $\inf_{W^{1,p}(\Omega)} I_{[-\infty,0]} = -\infty$.

Furthermore, it is easily seen that the non-trivial critical points of $I_{[-\infty,u_s]}$ different from u_s are sign-changing solutions of (P) by Lemma 15 because (P) has no negative solutions (refer to Remark 21). As a result, it is sufficient to obtain a non-trivial critical point of $I_{[-\infty,u_s]}$ different from u_s under the assumption that $I_{[-\infty,u_s]}$ has only finitely many critical points. By the same reasoning as in the proof of Theorem 1, since u_s is a $W^{1,p}(\Omega)$-local minimizer of $I_{[-\infty,u_s]}$, we may assume that there exists an $r > 0$ such that

$$d_0 := I_{[0,u_s]}(u_s) = I_{[-\infty,u_s]}(u_s) \leqslant I_{[-\infty,u_s]}(v) \quad \text{for every } v \in B(u_s,r), \tag{34}$$

$$I_{[-\infty,u_s]}(u_s) < \inf\bigl\{I_{[-\infty,u_s]}(v);\ v \in \partial B(u_s,r)\bigr\} =: d_1 < 0, \tag{35}$$

where $B(u_s,r) := \{v \in W^{1,p}(\Omega);\ \|u_s - v\| < r\}$. Here, we define the following:

$$\Gamma := \bigl\{\gamma \in C\bigl([0,1], W^{1,p}(\Omega)\bigr);\ \gamma(0) = u_s,\ I_{[-\infty,u_s]}\bigl(\gamma(1)\bigr) < d_0\bigr\},$$
$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_{[-\infty,u_s]}\bigl(\gamma(t)\bigr).$$

It is easily shown that $c \geqslant d_1$ provided $\Gamma \neq \emptyset$ because $\gamma([0,1]) \cap \partial B(u_s, r) \neq \emptyset$ for every $\gamma \in \Gamma$ by (34) and (35). Note that all the functionals $I_{[-\infty,u_s]}$, $I_{[-\infty,0]}$ and $I_{[0,u_s]}$ satisfy the Palais–Smale condition by Lemma 12 and Lemma 11. So, if $\Gamma \neq \emptyset$, then c is a critical value of $I_{[-\infty,u_s]}$ such that $c \geqslant d_1 > I_{[-\infty,u_s]}(u_s)$ (refer to the proof of Theorem 2 in [18]).

Next, we claim that there exists a path $\gamma_0 \in \Gamma$ satisfying $0 \notin \gamma_0([0,1])$, $u_s \notin \gamma((0,1])$ and $I_{[-\infty,u_s]}(\gamma_0(t)) \leqslant 0$ for every $t \in [0,1]$. In fact, because $I_{[-\infty,0]}$ has no non-trivial critical points (note that (P) has no negative solutions and $\inf_{W^{1,p}(\Omega)} I_{[-\infty,0]} = -\infty$), by applying the second deformation lemma to $I_{[-\infty,0]}$, we can obtain an $\tilde{\eta}_2 \in C([0,1], W^{1,p}(\Omega))$ such that $\tilde{\eta}_2(0) = -w_-$, $I_{[-\infty,0]}(\tilde{\eta}_2(1)) \leqslant d_0 - 1$ and $I_{[-\infty,0]}(\tilde{\eta}_2(t)) < I_{[-\infty,0]}(-w_-) \leqslant 0$ for every $0 < t \leqslant 1$, where $w \in C^1(\overline{\Omega})$ is the function described in the proof of Theorem 1 such that $\|w\|_\infty < \min\{\delta_4, \min_{\overline{\Omega}} u_s\}$. Hence, by replacing η_2 with $\tilde{\eta}_2$ as in the proof of Theorem 1, we can obtain a desired path $\gamma_0 \in \Gamma$.

By applying a similar argument to $I_{[-\infty,u_s]}$ instead of $I_{[u_l,u_s]}$ as in the proof of Theorem 1 (note $I_{[-\infty,u_s]}(\gamma_0(1)) < I_{[-\infty,u_s]}(u_s)$), we can prove that $(d_1 \leqslant) c < 0$ if $\gamma_0((0,1))$ contains no non-trivial critical points of $I_{[-\infty,u_s]}$. Consequently, we can get at least one non-trivial critical point of $I_{[-\infty,u_s]}$ different from u_s. So, this theorem is proved. $\square$

Proof of Theorem 5. By the same argument as in the proof of Theorem 1, we have a largest negative solution $u_l \in C^1(\overline{\Omega})$ of (P) being also a unique global minimizer of $I_{[u_l,0]}$ with $I_{[u_l,0]}(u_l) < 0$. Note that $I_{[u_l,+\infty]}$ and $I_{[0,+\infty]}$ are C^1 functionals satisfying the Palais–Smale condition by Lemma 12.

If (P) has at least one positive solution, then we can show the existence of a smallest positive solution and a sign-changing solution by a similar argument as in the proof of Theorem 4. Thus, we may assume that (P) has no positive solutions. Under this assumption, it suffices to show the existence of a non-trivial critical point of $I_{[u_l,+\infty]}$ different from u_l supposing that $I_{[u_l,+\infty]}$ has only finitely many critical points. Hence, by using u_l, $I_{[u_l,+\infty]}$, $I_{[0,+\infty]}$ and $I_{[u_l,0]}$ instead of u_s, $I_{[-\infty,u_s]}$, $I_{[-\infty,0]}$ and $I_{[0,u_s]}$ as in the case (b) of Theorem 4 respectively, we can obtain at least one non-trivial critical point of $I_{[u_l,+\infty]}$ other than u_l. $\square$

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Appendix A

For readers' convenience, we show the following strong maximum principle by Zhang's methods [21]. Here, we remark that we do not need an assumption concerning the dimension N and p although it is imposed in [21]. Moreover, we mention that Zhang's result [21] can not be applied to the nonlinear term $u^{p-1}(1 + |\log u|)$. In what follows, we adopt the notation like $\alpha/\beta\gamma\cdots\omega$ to express $\alpha/(\beta\gamma\cdots\omega)$.

Theorem A. *Assume that the map A satisfies the assumption (A)(i), (ii), (iii) and (v) for some $p \in (1,\infty)$. Let $x_1 \in \partial\Omega$, $\lambda \geqslant 0$ and $p \leqslant q < \infty$. If $u \in C^1(\Omega \cup \{x_1\})$ satisfies $u(x_1) = 0$, $u > 0$ in Ω and*

$$-\operatorname{div} A(x, \nabla u) + \lambda u^{q-1}\bigl(1 + |\log u|\bigr) \geqslant 0 \quad \textit{in } \Omega \ (\textit{in distribution sense}), \tag{36}$$

then $\partial u(x_1)/\partial n < 0$ holds.

Proof. In this proof, C_0, C_1 and C_3 denote the constants appearing in condition (A). We assume $N \geqslant 2$ because the proof is easier for $N = 1$. Because $\partial\Omega$ is of class C^2, we can choose $x_2 \in \Omega$ and $0 < T < \min\{1, C_0/4NC_3\}$ such that $B(x_2, 2T) \subset \Omega$ and $\partial B(x_2, 2T) \cap \partial\Omega = \{x_1\}$, where $B(x, T) := \{y \in \mathbb{R}^N;\ |x - y| < T\}$. Set $\rho_0 := \min\{u(x);\ |x - x_2| = T\} > 0$ (note $u > 0$ in Ω) and $t_1 := \min\{t_0, e^{-1/(q-1)}\}$, where $t_0 \in (0, 1]$ is the constant in assumption (A)(v). Choose a real number k such that

$$k > \max\left\{\frac{1}{T}, \frac{t_1}{2\rho_0}, \left(\frac{2}{t_1}\right)^{2(p-1)}, \left(\frac{8\lambda}{C_0(q-1)}\right)^2, \left(\frac{16\lambda T}{C_0}\right)^{1/(p-1)}, \frac{8NC_3}{t_1 C_0}, \frac{4(N-1)C_1}{T(p-1)C_0}\right\}.$$

Because of $4NC_3/C_0k^2 < t_1/2k$ and $1/k^{1+1/2(p-1)} < t_1/2k$, we can take a positive number ρ satisfying $\max\{\frac{4NC_3}{C_0k^2}, 1/k^{1+1/2(p-1)}\} < \rho < \frac{t_1}{2k}$ $(< \min\{\rho_0, 1/2\})$. Define

$$Y := \{x \in \Omega;\ T < |x - x_2| < 2T\} \quad \text{and} \quad v(t) := \rho\frac{e^{kt} - 1}{e^{kT} - 1} \quad \text{for } t \in [0, T].$$

Now, we construct a positive sub-solution w of (36) in Y with $u \geqslant w$ on ∂Y by using the function v. So, we define

$$w(x) := w(r) := v(2T - r) = v(t)$$

for $x \in Y$ with $r = |x - x_2|$ and $t = 2T - r$. It follows that $w \in C^\infty(\overline{Y})$ with $u \geqslant w$ on ∂Y, $w'(r) = -v'(t)$, $w''(r) = v''(t) = kv'(t) > 0$ and

$$(1 \geqslant)\ t_0 \geqslant t_1 > 2\rho k > \rho k\left(1 + \frac{1}{e^{kT} - 1}\right) \geqslant v'(t) > kv(t),$$

$$-\log v'(t) \leqslant \log\bigl((e^{kT} - 1)/\rho k\bigr) \leqslant kT + \log(1/\rho k) \leqslant kT + \frac{1}{\rho k} - 1 \leqslant \frac{C_0 k}{2NC_3} - 1,$$

$$-\log v(t) \leqslant kT + \frac{1}{q-1}\log\frac{1}{(\rho(e^{kt} - 1))^{(q-1)}} \leqslant kT + \frac{1}{(q-1)(\rho(e^{kt} - 1))^{(q-1)}}$$

(note $\log(x + 1) \leqslant x$). As a consequence of inequalities above, we first obtain the following estimate in Y:

$$\begin{aligned}
&-\frac{C_0 k}{4}\bigl(v'\bigr)^{p-1} + \lambda v^{q-1}\bigl(1 + |\log v|\bigr)\\
&\leqslant \bigl(v'\bigr)^{p-1}\left(-\frac{C_0 k}{4} + \lambda\rho^{q-p}k^{1-p}(1 + kT)\right) + \frac{\lambda}{(q-1)(e^{kT} - 1)^{q-1}}\\
&\leqslant \bigl(v'\bigr)^{p-1}k\bigl(-C_0/4 + 2\lambda k^{1-p}T\bigr) + \frac{\lambda}{(q-1)(e^{kT} - 1)^{p-1}}\\
&\leqslant \bigl(e^{kT} - 1\bigr)^{1-p}\left(-\frac{C_0 k}{8}(\rho k)^{p-1} + \frac{\lambda}{q-1}\right)\\
&\leqslant \bigl(e^{kT} - 1\bigr)^{1-p}\left(-\frac{C_0 k}{8}k^{-1/2} + \frac{\lambda}{q-1}\right) < 0. \qquad (37)
\end{aligned}$$

Note $\rho < 1 < \min\{p, kT\}$ and so $\lambda\rho^{q-p}k^{1-p}(1+kT) < 2\lambda k^{2-p}T$ and note $\rho k > k^{-1/2(p-1)}$. Next, we can obtain the following estimate in Y:

$$\begin{aligned}
-\operatorname{div} A(x, \nabla w) &= -\frac{w'}{r}\sum_{i=1}^{N}\partial_{x_i} a\bigl(x, -w'\bigr)\bigl(x_i - (x_2)_i\bigr) + w'w''\partial_t a\bigl(x, -w'\bigr) \\
&\quad - \bigl(w'' + (N-1)w'/r\bigr)a\bigl(x, -w'\bigr) \\
&\leqslant NC_3\bigl(v'\bigr)^{p-1}\bigl(-\log v'\bigr) - kC_0\bigl(v'\bigr)^{p-1} + (N-1)a\bigl(x, v'\bigr)v'/r \\
&\leqslant \bigl(v'\bigr)^{p-1}\bigl(NC_3\bigl(-\log v'\bigr) - kC_0 + (N-1)C_1/T(p-1)\bigr) \\
&\leqslant \bigl(v'\bigr)^{p-1}\bigl(-kC_0/2 - NC_3 + (N-1)C_1/T(p-1)\bigr) < -\frac{C_0 k}{4}\bigl(v'\bigr)^{p-1}. \qquad (38)
\end{aligned}$$

Here, we use the following inequalities for $x \in Y$, $t_0 > t > 0$, $s > 0$;

$$\begin{gathered}
\partial_{x_j} a(x, t)t = D_x A(x, te_j)e_j \cdot e_j \leqslant C_3 t^{p-1}(-\log t), \\
\partial_t a(x, s)s + a(x, s) = D_y A(x, se_1)e_1 \cdot e_1 \geqslant C_0 s^{p-2} \quad \text{and} \\
a(x, s)s = A(x, se_1) \cdot e_1 \leqslant C_1 s^{p-1}/(p-1),
\end{gathered}$$

where $\{e_j\}_{1 \leqslant j \leqslant N}$ denotes the usual orthonormal basis in $\mathbb{R}^N$. These inequalities are easy consequences of (A)(v), (A)(iii) and Remark 7, respectively. The inequalities (37) and (38) imply that w satisfies

$$-\operatorname{div} A(x, \nabla w) + \lambda w^{q-1}\bigl(1 + |\log w|\bigr) < 0 \quad \text{in } Y \qquad (39)$$

with $u \geqslant w$ on ∂Y. Note that the function $g(t) := t^{q-1}|\log t|$ is strictly increasing on the interval $(0, \min\{1, e^{-1/(q-1)}\})$ and also that $w(r) < t_1 \leqslant \min\{1, e^{-1/(q-1)}\}$. Since the map A is strictly monotone in the second variable and $(w-u)_+ \in W_0^{1,p}(Y)$, we have that $u(x) \geqslant w(x)$ for every $x \in Y$ by the following inequality (take $(w-u)_+$ as test function (note $u \in C^1(\overline{Y})$ and $w \in C^\infty(\overline{Y})$) and use (36) and (39)):

$$\begin{aligned}
0 &\leqslant \int_{\{x \in Y;\ w(x) > u(x)\}} \bigl(A(x, \nabla w) - A(x, \nabla u)\bigr)(\nabla w - \nabla u)\, dx \\
&\leqslant -\lambda \int_Y \bigl(w^{q-1} - u^{q-1}\bigr)(w-u)_+\, dx - \lambda \int_Y \bigl(g(w) - g(u)\bigr)(w-u)_+\, dx \leqslant 0.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
-2T\frac{\partial u}{\partial n}(x_1) &= \lim_{s \to +0} \frac{u(x_1 + s(x_2 - x_1)) - u(x_1)}{s} \\
&\geqslant \lim_{s \to +0} \frac{w(x_1 + s(x_2 - x_1)) - w(x_1)}{s} = 2Tv'(0) > 0
\end{aligned}$$

since $u(x_1) = 0$, $u \in C^1(\Omega \cup \{x_1\})$ and $n(x_1) = -(x_2 - x_1)/2T$. $\square$

By examining the argument in the proof of Theorem A, we can replace Ω with $\Omega \cap \Omega_1$ for some neighborhood Ω_1 of x_1 in the hypothesis of Theorem A, and it suffices that Ω satisfies the interior-ball condition at $x_1 \in \partial\Omega$ instead of C^2 boundary $\partial\Omega$. Next, by using Theorem A, we can show the following result. It is used to obtain a positive solution of (P).

Theorem B. *Assume that the map A satisfies the assumption (A)*(i), (ii), (iii) *and* (v) *for some $p \in (1,\infty)$. Let $\lambda \geqslant 0$ and $p \leqslant q < \infty$. If $u \in C^1(\Omega)$ satisfies $u \geqslant 0$ in Ω, $u \not\equiv 0$ in Ω and*

$$-\operatorname{div} A(x,\nabla u) + \lambda u^{p-1}\bigl(1+|\log u|\bigr) \geqslant 0 \quad \textit{in } \Omega \ (\textit{in distribution sense}), \tag{40}$$

then $u > 0$ in Ω. (As usual, $u^{p-1}|\log u|$ should be interpreted as 0 at a point where $u = 0$.)

Proof. We set $[u>0] := \{x \in \Omega;\ u(x) > 0\}$, and then we note that $[u>0] \neq \emptyset$ by our assumption. Since Ω is connected, it suffices to show $\partial[u>0] \cap \Omega = \emptyset$. Thus, by way of contradiction, we assume that $\partial[u>0] \cap \Omega \neq \emptyset$. Then, we may assume that there exists at least one point $x_1 \in \partial[u>0] \cap \Omega$ such that $[u>0]$ satisfies the interior-ball condition at x_1. Indeed, since the set K of all $x \in \partial[u>0]$ such that $[u>0]$ satisfies the interior-ball condition at x is dense in $\partial[u>0]$ (note that $[u>0]$ is a bounded open set and refer to the following lemma), $\partial[u>0] \cap \Omega = \emptyset$ holds provided $K \cap \Omega = \emptyset$. Hence, there exist $x_1 \in \partial[u>0] \cap \Omega$, $x_2 \in [u>0]$ and $0 < T < \min\{1, C_0/4NC_3\}$ such that $B(x_2, 2T) \subset [u>0]$ and $\partial B(x_2, 2T) \cap \partial[u>0] = \{x_1\}$. Because $\min\{u(x);\ |x - x_2| = T\} > 0$, $u(x_1) = 0$ and $-\operatorname{div} A(x,\nabla u) + \lambda u^{p-1}(1+|\log u|) \geqslant 0$ in $B(x_2, 2T)$, by the same argument as in Theorem A, $\partial u/\partial n(x_1) \neq 0$ with $n(x_1) = -(x_2 - x_1)/2T$. On the other hand, since $x_1 \in \Omega$ is a global minimizer of u (note that $u(x_1) = 0 \leqslant u(x)$ for every $x \in \Omega$), we have $\nabla u(x_1) = 0$. This is a contradiction. Therefore, $\partial[u>0] \cap \Omega = \emptyset$ holds. $\square$

For readers' convenience, we state the following elementary fact.

Lemma. *Let O be a non-empty bounded open subset of $\mathbb{R}^N$, and let K be the set of all $x \in \partial O$ such that O satisfies the interior-ball condition at x. Then, K is dense in ∂O.*

Proof. Fix any $x \in \partial O$ and $\varepsilon > 0$. Then, we can choose $x_\varepsilon \in O$ satisfying $|x - x_\varepsilon| < \varepsilon$. Since ∂O is compact and $x_\varepsilon \notin \partial O$, there exists $y_\varepsilon \in \partial O$ such that $0 < |x_\varepsilon - y_\varepsilon| = \operatorname{dist}(x_\varepsilon, \partial O) < \varepsilon$. It is easily verified that O satisfies the interior-ball condition at y_ε, that is, $y_\varepsilon \in K$. Therefore, our conclusion holds because $x \in \partial O$ and ε are arbitrary (note $|x - y_\varepsilon| < 2\varepsilon$). $\square$

Finally, we show the following fact which is needed to study the regularity of a solution.

Theorem C. *Suppose that the map A satisfies the assumption (A)*(i), (ii) *and* (iii) *for some $p \in (1,\infty)$. Assume that $u_0 \in C^1(\overline{\Omega})$ and f is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying the subcritical growth condition (g). Let h be a Carathéodory function on $\Omega \times \mathbb{R}$ such that $\int_\Omega h(x,u)\varphi\,dx$ is well defined for every $u \in W^{1,p}(\Omega)$, $\varphi \in L^\infty(\Omega)$ and $h(x,t)t \geqslant 0$ for every $t \in \mathbb{R}$ and a.e. $x \in \Omega$. We set $\tilde{A}(x,y) := A(x, \nabla u_0(x) + y) - A(x, \nabla u_0(x))$. Moreover, supposing that μ is a non-positive real number, consider the following equation*:

$$-\operatorname{div} \tilde{A}(x,\nabla u) = f(x, u + u_0) + \mu h(x,u) \quad \textit{in } \Omega, \qquad \frac{\partial u}{\partial n} = 0 \quad \textit{on } \partial\Omega. \tag{41}$$

Then, every solution $u \in W^{1,p}(\Omega)$ to (41) *belongs to $L^\infty(\Omega)$, and there exist constants $C > 0$ and $q \geqslant 1$ independent of u, μ and h, for which $\|u\|_\infty \leqslant C \max\{1, \|u\|^q\}$ holds.*

Proof. Let $r \in [p, p^*)$ be a constant for which (g) is satisfied. First, it readily follows that there exists a constant $C > 0$ such that

$$\tilde{A}(x, y)y = \big(A\big(x, \nabla u_0(x) + y\big) - A\big(x, \nabla u_0(x)\big)\big)\big(\big(\nabla u_0(x) + y\big) - \nabla u_0(x)\big) \geqslant \begin{cases} C|y|^2(|\nabla u_0(x) + y| + |\nabla u_0(x)|)^{p-2} & \text{or} \\ C|y|^p \end{cases} \tag{42}$$

for every $x \in \Omega$ and $y \in \mathbb{R}^N$, according as $1 < p < 2$ and $|\nabla u_0(x) + y| + |\nabla u_0(x)| \neq 0$ or $p \geqslant 2$ (refer to [13, Lemma 2.1]).

Now, for $\varphi = u_+$ or u_- and $M > 0$, we put $\varphi_M(x) := \min\{\varphi(x), M\}$ to simplify the notation. Then, because we have

$$(q+1)\int_\Omega \varphi_M^q \tilde{A}(x, \pm\nabla\varphi_M)(\pm\nabla\varphi_M)\,dx = \pm\int_\Omega \tilde{A}(x, \nabla u)\nabla\big(\varphi_M^{q+1}\big)\,dx \leqslant \pm\int_\Omega f(x, u_0 + u)\varphi_M^{q+1}\,dx$$

for $\varphi = u_\pm$, respectively (note $\mu \leqslant 0 \leqslant h(x,t)t$ and $\nabla(\varphi_M^{q+1}) = (q+1)\varphi_M^q\nabla\varphi_M$) by taking φ_M^{q+1} as test function in (41), we obtain

$$\int_\Omega \big|f(x, u_0 + u)\big|\varphi_M^{q+1}\,dx \geqslant \begin{cases} (q+1)C\int_\Omega |\nabla\varphi_M|^p(\varphi_M)^q\,dx \\ \quad (\text{if } p \geqslant 2, \text{ or } 1 < p < 2 \text{ and } \nabla u_0 \equiv 0), \\ (q+1)C\int_\Omega |\nabla\varphi_M|^2(2\|\nabla u_0\|_\infty + |\nabla\varphi_M|)^{p-2}\varphi_M^q\,dx \\ \quad (\text{if } 1 < p < 2 \text{ and } \nabla u_0 \not\equiv 0), \end{cases} \tag{43}$$

for every $q > 0$ and $M > 0$ by (42). In particular, in the case of $1 < p < 2$ and $\nabla u_0 \not\equiv 0$, we obtain the following inequality for every $q > 0$ whenever $\varphi \in L^{q+r}(\Omega)$, by a number of elementary but careful applications of Hölder's inequality, (g) and (43):

$$\begin{aligned} C_*\|\varphi_M\|_{p^*q'}^{p+q} &= C_*\big\|(\varphi_M)^{q'}\big\|_{p^*}^p \\ &\leqslant \big\|(\varphi_M)^{q'}\big\|^p = q'^p\int_\Omega |\nabla\varphi_M|^p(\varphi_M)^q\,dx + \|\varphi_M\|_{p+q}^{p+q} \\ &\leqslant q'^p\int_{|\nabla\varphi_M|\leqslant 1}(\varphi_M)^q\,dx + q'^p\int_{|\nabla\varphi_M|>1}|\nabla\varphi_M|^p(\varphi_M)^q\,dx + \|\varphi_M\|_{p+q}^{p+q} \\ &\leqslant q'^p C\max\big\{1, \|\varphi\|_{q+r}^{q+r}\big\} \end{aligned} \tag{44}$$

where $q' = 1 + q/p$, $\overline{p}^* = p^*$ if $N > p$, and in the case of $N \leqslant p$, $\overline{p}^* > r$ is an arbitrarily fixed constant. Moreover, C_* and $C \geqslant C_*$ are constants independent of u, q, M, λ and h (note $u_0 \in C^1(\overline{\Omega})$, of which C_* comes from the continuous embedding of $W^{1,p}(\Omega)$ into $L^{\overline{p}^*}(\Omega)$. Similarly, in the other cases, the same form of inequality $C_* \|\varphi_M\|_{\overline{p}^* q'}^{p+q} \leqslant (q')^p C \max\{1, \|\varphi\|_{q+r}^{q+r}\}$ holds provided $\varphi \in L^{q+r}(\Omega)$. Define a sequence $\{q_m\}_m$ by $q_0 := \overline{p}^* - r$ and $q_{m+1} := \overline{p}^*(p + q_m)/p - r$. Then, we see that $\varphi \in L^{\overline{p}^*(p+q_m)/p}(\Omega) = L^{r+q_{m+1}}(\Omega)$ holds if $\varphi \in L^{r+q_m}(\Omega)$ by applying Fatou's lemma to (44) and letting $M \to \infty$. Here, we also see $q_{m+1} = \overline{p}^* q_m/p + \overline{p}^* - r \geqslant (\overline{p}^*/p)^{m+1} q_0 \to \infty$ as $m \to \infty$. And if we set $b_m := \max\{1, \|\varphi\|_{r+q_m}^{r+q_m}\}$, then it follows from (44) that

$$\begin{aligned}
\log b_m &\leqslant \frac{r+q_m}{p+q_{m-1}} \bigl(\log b_{m-1} + p \log\bigl(C(p+q_{m-1})/C_*\bigr)\bigr) \\
&\leqslant P^{-m} \log b_0 + p \sum_{i=1}^{m} P^{-i} \log\bigl(C(p+q_{m-i})/C_*\bigr) \\
&\leqslant P^{-m} \log b_0 + p \sum_{i=1}^{m} P^{-i} \log\bigl(C\bigl(p + P^{-m+i}(m-i+1)q_0\bigr)/C_*\bigr)
\end{aligned}$$

where $P = p/\overline{p}^* = (p+q_{m-1})/(r+q_m) < 1$ (note $q_{m+1} = P^{-1} q_m + q_0$). So, we have

$$\begin{aligned}
&\log \max\bigl\{1, \|\varphi\|_{r+q_m}\bigr\} \\
&\quad = \frac{\log b_m}{r+q_m} = \frac{(P^{-1}-1)\log b_m}{(P^{-1}-1)r + q_0(P^{-m-1}-1)} \\
&\quad \leqslant \frac{(P^{-m-1}-P^{-m})\log b_0}{(P^{-1}-1)r + q_0(P^{-m-1}-1)} + \frac{p}{q_0} \sum_{k=0}^{m-1} P^k \log\bigl(CP^{-k}\bigl(p+(k+1)q_0\bigr)/C_*\bigr) \\
&\quad \leqslant \frac{(1-P)(r+q_0)\log(C'\max\{1, \|\varphi\|\})}{(P^m - P^{m+1})r + q_0(1-P^{m+1})} + C' \sum_{k=0}^{\infty} P^k (k+1) < +\infty,
\end{aligned}$$

where $C' \geqslant 1$ is a constant independent of u, m, μ and h (note that $\log(k+1) \leqslant k$). This inequality implies that $u \in L^\infty(\Omega)$ and $\|u\|_\infty \leqslant C \max\{1, \|u\|\}^{(\overline{p}^*-p)/(\overline{p}^*-r)}$ holds for some positive constant C independent of u, μ and h because of $\|u_\pm\|_\infty = \lim_{m\to\infty} \|u_\pm\|_{r+q_m}$ and

$$\lim_{m\to\infty} \frac{(1-P)(r+q_0)}{(P^m - P^{m+1})r + q_0(1-P^{m+1})} = \frac{(1-P)(r+q_0)}{q_0} = \frac{\overline{p}^* - p}{\overline{p}^* - r}. \qquad \square$$

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